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THE INVERSION OF FRACTIONAL INTEGRALS ON A SPHERE

BY

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ABSTRACT

The purpose of the paper is to invert Riesz potentials and some other fractional integrals on the n-dimensional spherical surface in $\mathbb{R}^{n+1}$ in the closed form. New descriptions of spaces of the fractional smoothness on a sphere are obtained in terms of spherical hypersingular integrals. It is shown that Riesz potentials of the orders $n, n+2, n+4, \ldots$ on a sphere are Noether operators and their d-characteristic depends on the radius of the sphere.

Introduction

Fractional integrals on the surface of the n-dimensional unit sphere $\Sigma_n \subset \mathbb{R}^{n+1}$ may be defined in a large number of ways (see, e.g., [15]). We consider a Riesz potential

$$(1) \qquad (I^\alpha \varphi)(x) = c_{n,\alpha} \int_{\Sigma_n} |x-y|^{\alpha-n} \varphi(y) dy,$$

where $\alpha > 0$; $\alpha \neq n, n+2, n+4, \ldots$;

$$(2) \qquad c_{n,\alpha} = 2^{-\alpha} \pi^{-n/2} \Gamma\left(\frac{n-2}{2}\right) / \Gamma\left(\frac{\alpha}{2}\right).$$

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Due to the outward simplicity and to the plurality of applications the Riesz potential is a typical object in fractional calculus. Nevertheless, the inversion method for I^α, covering all admissible α seems unknown. There is a simple idea to change variables in (1), using the stereographic projection, and to turn the potential (1) in such a way into the Riesz potential over $\mathbb{R}^n$ (up to some multipliers). The latter may be inverted by diverse known methods (see [14], [13]). This approach, suggested by the author, enables us to obtain a number of estimates of $I^\alpha\varphi$ using the corresponding estimates of the space potentials (see [10], [19]). Nevertheless, this way leads to the unnatural awkward construction of $(I^\alpha)^{-1}$ which depends on the pole of the projection. Furthermore, the proof of such an inversion formula is connected with large technical difficulties. It is more preferable to construct the operator $(I^\alpha)^{-1}$ directly in spherical terms. In [10] Pavlov P.M. and Samko S.G. proved that if $f = I^\alpha\varphi$, $\varphi \in L_p(\Sigma_n)$, $0 < \alpha < 2$, $1 \le p < \infty$, then

$$\varphi(x) = c_1 f(x) + c_2 \int_{\Sigma_n} \frac{f(x) - f(y)}{|x-y|^{n+\alpha}} dy, \tag{3}$$

where

$$c_1 = \Gamma\left(\frac{n+\alpha}{2}\right) / \Gamma\left(\frac{n-\alpha}{2}\right),$$

$$c_2 = \frac{2^{\alpha-1}\alpha\Gamma\left(\frac{n+\alpha}{2}\right)}{\pi^{n/2}\Gamma\left(1-\frac{\alpha}{2}\right)},$$

$$\int_{\Sigma_n}(\ldots) = \overset{(L_p)}{\lim_{\varepsilon\to\infty}} \int_{|x-y|>\varepsilon}(\ldots).$$

The method of [10] gives no answer how to invert I^α for all $\alpha \ge 2$. In the present paper we suggest two different inversion methods for Riesz potentials of finite Borel measures in spherical terms. These methods are suitable for all $\alpha > 0$ (the definition of $I^\alpha\varphi$ for $\alpha = n, n+2, n+4, \ldots$, see below) and may be generalized for all complex α with $\operatorname{Re}\alpha > 0$ as in [13]. Our formulas contain hypersingular integrals, the convergence of which is associated with a type of the measure to be restored. For arbitrary finite Borel measure these integrals converge in a weak sense. If the measure is absolutely continuous with a density belonging to $L_p(\Sigma_n)$, $1 \le p < \infty$, then the convergence of hypersingular integrals is treated in the "almost everywhere" sense and in L_p-norm. If the density is continuous, then a uniform convergence is used.

In section 1, we construct the operator $(I^\alpha)^{-1}$ using a direct regularization of the potential $I^\alpha\varphi$. This method was suggested by A.Marchaud in [8] for one-dimensional fractional integrals and was developed in [13] for multidimensional potentials. The case $\alpha = n$, when $I^\alpha\varphi$ turns into the logarithmic potential, is considered in section 2. Another inversion method for $I^\alpha\varphi$, based on properties of a Poisson integral, is given in section 3.

The inversion problem for potentials (1) is closely connected with the characterization of functions of a fractional smoothness on a sphere. In section 4 we give a number of diverse descriptions of the spaces $L_p^\alpha(\Sigma_n)$, $C^\alpha(\Sigma_n)$, $M^\alpha(\Sigma_n)$ generated by L_p-functions, by continuous functions and by finite Borel measures respectively. By the way we obtain inversion formulas for some fractional integral operators introduced by du Plessis N.[11], Greenwald H.C. [6, 7], Muckenhoupt B. and Stein E.M.[9]. All these operators have the same range as I^α (with the exception of some values of α) and are built by means of a Poisson integral.

The investigation of Riesz potentials of the orders $\alpha = n + 2k$, $k = 0, 1, \ldots$, leads to the following integral equation on a sphere $\Sigma_n(a) = \{x \in \mathbf{R}^{n+1} : |x| = a\}$:

$$\int_{\Sigma_n(a)} \varphi(y)|x-y|^{2k} \log|x-y|dy = f(x). \tag{4}$$

In section 5 we show that in contrast to the case $\alpha \neq n+2k$ the operator in the left-hand side of (4) may be the Noether one with a nontrivial d-characteristic. We construct its two-sided regularizer and the d-characteristic explicity. It is interesting that the d-characteristic depends on the value of a radius a.

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Notation:

$$\Sigma_n = \{x \in \mathbf{R}^{n+1} : |x| = 1\}, \ \sigma_n = |\Sigma_n| = 2\pi^{(n+1)/2}/\Gamma\left(\frac{n+1}{2}\right);$$

dx denotes the Lebesgue measure on Σ_n; $\mathcal{Y}(\Sigma_n) = \{Y_{m,\mu}(x)\}$ denotes a complete orthonormal system of spherical harmonics on Σ_n; $m = 0, 1, \ldots$; $\mu = 1, 2, \ldots, d_n(m)$, $d_n(m)$ being a dimension of the subspace of harmonics of the order m, $d_n(m) = (n+2m-1)\frac{(n+m-2)!}{m!(n-1)!}$ (see [18]). $\mathcal{B}(\Sigma_n)$ is the Borel σ-algebra of Σ_n. $M(\Sigma_n)$ denotes a Banach space of all regular complex valued finite Borel measures on $\mathcal{B}(\Sigma_n)$ with the norm $\|\nu\|_M$ equaled to a total variation of the measure ν on Σ_n ([3]); $C(\Sigma_n)$ denotes the space of all continuous functions on Σ_n; $S(\Sigma_n)$ denotes the space of all infinitely differentiable functions on Σ_n with the standard Shwartz topology; $S'(\Sigma_n)$ is a dual to $S(\Sigma_n)$; (f,ω) denotes a value of a functional $f \in S'(\Sigma_n)$ on a function $\omega \in S(\Sigma_n)$. If $f \in M(\Sigma_n)$ $(f \in L_1(\Sigma_n))$, then

$$(f,\omega) = \int_{\Sigma_n} \omega(x) df \qquad \left((f,\omega) = \int_{\Sigma_n} \omega(x) f(x) dx\right);$$

$f_{m,\mu} = (f, Y_{m,\mu})$ denote Fourier-Laplace coefficients of a functional $f \in S'(\Sigma_n)$; $e_{n+1}(0,\ldots,0,1)$; $a_+^\lambda = (\sup\{a,0\})^\lambda$; $P^{(\rho,\sigma)}(t)$ denotes a Jacobi polynomial; $\mathbb{Z}_+$ denotes the set of all nonnegative integers;

$$\|\varphi\|_p = \|\varphi\|_{L_p(\Sigma_n)};$$

$$P_r(x,y) = \frac{1-r^2}{\sigma_n |y - rx|^{n+1}} \text{ is a Poisson kernel, } 0 < r < 1;$$

$f(x,r) = (f, P_r(x,\cdot))$ denotes a Poisson integral of a function (measure) f.

$$(I_+^\lambda \psi)(\tau) = \frac{1}{\Gamma(\lambda)} \int_{-\infty}^{\tau} \psi(t)(\tau - t)^{\lambda-1} dt \tag{5}$$

is a Riemann-Liouville fractional integral of the order $\lambda > 0$. We define a truncated Marchaud derivative by the equality

$$(D_{+,\varepsilon}^\lambda \psi)(\tau) = \frac{1}{\kappa_\ell(\lambda)} \int_\varepsilon^\infty \left(\sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j f(\tau - jt) \right) \frac{dt}{t^{1+\lambda}},$$

where $\varepsilon > 0$, $\ell > \lambda$,

$$\kappa_\ell(\lambda) = \int_0^\infty \frac{(1-e^{-t})^\ell}{t^{1+\lambda}} dt$$

(see [14]).

Let $E \subset \mathbb{R}$ be some set with a limit point ε_0, and let $\{A_\varepsilon\}_{\varepsilon \in E}$ be a family of linear operators defined on $\mathcal{Y}(\Sigma_n)$. If $\lim_{\varepsilon \to \varepsilon_0} A_\varepsilon Y_{m,\mu} = Y_{m,\mu}$ $\forall Y_{m,\mu} \in \mathcal{Y}(\Sigma_n)$, then the famlily $\{A_\varepsilon\}$ will be called an approximate identity as $\varepsilon \to \varepsilon_0$.

Let us introduce functional spaces to be used later. Given $\alpha \in \mathbb{R}$, $1 \le p \le \infty$, we denote by $L_p^\alpha(\Sigma_n)$ $(C^\alpha(\Sigma_n), M^\alpha(\Sigma_n))$ the space of functionals $f \in S'(\Sigma_n)$ with the following property: for each $f \in S'(\Sigma_n)$ there exists a function $f^{(\alpha)} \in L_p(\Sigma_n)$

($f^{(\alpha)} \in C(\Sigma_n)$, a measure $f^{(\alpha)} \in M(\Sigma_n)$) such that $f^{(\alpha)}_{m,\mu} = (m+1)^\alpha f_{m,\mu}$ for any m, μ. The space $L^\alpha_p(\Sigma_n)$ ($C^\alpha(\Sigma_n), M^\alpha(\Sigma_n)$) is a Banach one with respect to the norm

$$\|f\| = \|f^{(\alpha)}\|_p \qquad (\|f\| = \|f^{(\alpha)}\|_{C(\Sigma_n)}, \ \|f\| = \|f^{(\alpha)}\|_{M(\Sigma_n)}). \tag{6}$$

If $\alpha > 0$ the elements of the spaces $L^\alpha_p(\Sigma_n)$, $C^\alpha(\Sigma_n)$, $M^\alpha(\Sigma_n)$ are ordinary functions represented by spherical fractional integrals (see section 4). Besides the Riesz potential with the expansion

$$I^\alpha \varphi \sim \sum_{m,\mu} \frac{\Gamma\left(m + \frac{n-\alpha}{2}\right)}{\Gamma\left(m + \frac{n+\alpha}{2}\right)} \varphi_{m,\mu} Y_{m,\mu} \tag{7}$$

(see [15]) we use the following fractional integrals:

$$I^\alpha_1 \varphi = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\rho)^{\alpha-1} \varphi(x,\rho) d\rho \qquad \left(\sim \sum_{m,\mu} \frac{\Gamma(m+1)}{\Gamma(m+1+\alpha)} \varphi_{m,\mu} Y_{m,\mu}\right), \tag{8}$$

$$I^\alpha_2 \varphi = \frac{1}{\Gamma(\alpha)} \int_0^1 \left(\log \frac{1}{\rho}\right)^{\alpha-1} \varphi(x,\rho) d\rho \qquad \left(\sim \sum_{m,\mu} (m+1)^{-\alpha} \varphi_{m,\mu} Y_{m,\mu}\right), \tag{9}$$

$$I^\alpha_3 \varphi = \frac{1}{\Gamma(\alpha)} \int_0^1 \left(\log \frac{1}{\rho}\right)^{\alpha-1} \varphi(x,\rho) \frac{d\rho}{\rho} \qquad \left(\sim \sum_{m,\mu} m^{-\alpha} \varphi_{m,\mu} Y_{m,\mu}\right), \tag{10}$$

$$I^\alpha_4 \varphi = \frac{\pi^{1/2}(n-1)^{(1-\alpha)/2}}{\Gamma(\alpha/2)} \cdot \int_0^1 \rho^{(n-3)/2} \left(\log \frac{1}{\rho}\right)^{\alpha-1} I_{(\alpha-1)/2}\left(\frac{n-1}{2} \log \frac{1}{\rho}\right) \varphi(x,\rho) d\rho$$
$$\left(\sim \sum_{m,\mu} (m(m+n-1))^{-\alpha/2} \varphi_{m,\mu} Y_{m,\mu}\right), \tag{11}$$

$\varphi(x,\rho)$ being a Poisson integral of a function (measure) φ, $I_{(\alpha-1)/2}(z)$ being a modified Bessel function of the first kind. The expansions above may be easily obtained by means of well known expansion of a Poisson integral

$$\varphi(x,\rho) \sim \sum_{m,\mu} \rho^m \varphi_{m,\mu} Y_{m,\mu}(x).$$

The integral (8) was introduced in [11]. The expansions (9), (10) and (11) were considered in [6]-[7], [9] and [1] respectively (see also [15], [2]). The mean value of φ on Σ_n is supposed to be zero in (10), (11). We denote by $\mathring{L}_p(\Sigma_n)$, $\mathring{C}(\Sigma_n)$, $\mathring{M}(\Sigma_n)$ the subspaces of $L_p(\Sigma_n)$, $C(\Sigma_n)$, $M(\Sigma_n)$ respectively, consisting of functions(measures) with a zero mean value. It will be convenient to use the following notation:

$$X_p(\Sigma_n) = \begin{cases} L_p(\Sigma_n) & \text{if } 1 \le p < \infty, \\ C(\Sigma_n) & \text{if } p = \infty, \end{cases} \qquad X_p^\alpha(\Sigma_n) = \begin{cases} L_p^\alpha(\Sigma_n) & \text{if } 1 \le p < \infty, \\ C^\alpha(\Sigma_n) & \text{if } p = \infty. \end{cases}$$

∎ denotes the end of the proof.

1. The inversion of Riesz potentials by the direct regularization method

According to (7) in order to construct the operator $(I^\alpha)^{-1}$ we may continue $I^\alpha\varphi$ analytically to the half-plane $\Re\alpha < 0$ and then replace α by $-\alpha$. To do this we represent $I^\alpha\varphi$ as a one-dimensional integral with the extracted singularity in the integrand. Let us go over to the "polar coordinates" on a sphere by means of the formula

$$\begin{aligned} \int_{\Sigma_n} a(xy)\varphi(y)dy &= \sigma_{n-1}\int_0^\pi a(\cos\theta)(\sin\theta)^{n-1}(M^0_{\cos\theta}\varphi)(x)d\theta \\ &= \sigma_{n-1}\int_{-1}^1 a(t)(M_t^0\varphi)(x)(1-t^2)^{n/2-1}dt, \end{aligned} \tag{1.1}$$

where

$$\begin{aligned} (M_t^0\varphi)(x) &= \frac{(1-t^2)^{(1-n)/2}}{\sigma_{n-1}}\int_{xy=t}\varphi(y)dy \\ &= \sum_{m,\mu}\frac{m!\Gamma(n/2)}{\Gamma(m+n/2))}P_m^{(n/2-1,n/2-1)}(t)\varphi_{m,\mu}Y_{m,\mu}(x) \end{aligned} \tag{1.2}$$

is a mean value of φ on a planar section $\{y \in \Sigma_n : xy = t\}$ (see, e.g., [16], p. 183). By virtue of (1.1) we have

$$\begin{aligned} (I^\alpha\varphi)(x) &= 2^{(\alpha-n)/2}c_{n,\alpha}\int_{\Sigma_n}(1-xy)^{(\alpha-n)/2}\varphi(y)dy \\ &= \frac{2^{1-(\alpha+n)/2}\Gamma(\frac{n-\alpha}{2})}{\Gamma(n/2)\Gamma(\alpha/2)}\int_0^2 \eta^{\alpha/2-1}g_{x,\varphi}(1-\eta)d\eta, \end{aligned} \tag{1.3}$$

where

$$(1.4)\qquad g_{x,\varphi}(\tau)=(1+\tau)_+^{n/2-1}(M_\tau^0\varphi)(x).$$

Following to A.Marchaud's method ([8], [13]) we represent the analytical continuation of the integral (1.3) in the form of a difference integral. After replacing α by $-\alpha$ we obtain a solution of the equation $I^\alpha\varphi=f$ in the form

$$(1.5)\qquad \varphi(x)=\frac{1}{\gamma_\ell(\alpha)}\int_0^2\eta^{-\alpha/2-1}\left(\sum_{j=0}^{\ell}\binom{\ell}{j}(-1)^j g_{x,f}(1-j\eta)\right)d\eta \overset{\text{def}}{=} T^\alpha f,$$

where

$$\ell>\alpha/2,\quad \gamma_\ell(\alpha)=\kappa_\ell(\alpha/2)\Gamma(n/2)2^{(n-\alpha)/2-1}/\Gamma\left(\frac{n+\alpha}{2}\right),$$

$$(1.6)\qquad \kappa_\ell(\alpha/2)=\int_0^\infty(1-e^{-t})^\ell\frac{dt}{t^{\alpha/2+1}}$$

or

$$(1.7)\quad \varphi(x)=\frac{1}{\gamma_\ell(\alpha)}\int_0^2\eta^{-\alpha/2-1}\left[\sum_{j=0}^{\ell}\binom{\ell}{j}(-1)^j(2-j\eta)_+^{n/2-1}(M_{1-j\eta}^0 f)(x)\right]d\eta.$$

The integral in (1.7) may be transformed into an integral over Σ_n. Denote by ω_x some rotation with the property $x=\omega_x e_{n+1}$. Given $y\in\Sigma_n$, we write $y=(\eta,\sigma)$ if $y=(1-\eta)e_{n+1}+\sigma\sqrt{\eta(2-\eta)}$, $\eta\in[0,2]$, $\sigma\in\Sigma_{n-1}$. For $y=(\eta,\sigma)$, $j\in\mathbb{Z}_+$, $j\eta\le 2$ we denote $y_j=(j\eta,\sigma)$. The point $y_j\in\Sigma_n$ has the same "angle" coordinate as y, and its distance to e_{n+1} "along the vertical" is j times larger than the similar distance of the point y. Using this notation we can rewrite (1.7) in the following form

$$(1.7')\,\varphi(x)=\frac{1}{\gamma_\ell(\alpha)}\int_{\Sigma_n}\left(\sum_{j=0}^{\ell}\binom{\ell}{j}(-1)^j\left(\frac{2-j(1-y_{n+1})}{1+y_n}\right)_+^{n/2-1}f(\omega_x y_j)\right)\frac{dy}{(1-y_{n+1})^{(n+\alpha)/2}}.$$

One can show that $(1.7')$ coincides with (3) if $0<\alpha<2,\ \ell=1$.

To give a strict proof of (1.7), (1.7′) we introduce the truncated integral

$$(1.8)\qquad \begin{aligned}&(T_\varepsilon^\alpha f)(x)\\ &= \frac{1}{\gamma_\ell(\alpha)}\int_\varepsilon^2 \eta^{-\alpha/2-1}\left[\sum_{j=0}^{\ell}\binom{\ell}{j}(-1)^j(2-j\eta)_+^{n/2-1}(M_{1-j\eta}^0 f)(x)\right]d\eta\\ &= \frac{1}{\gamma_\ell(\alpha)}\int_{y_{n+1}<1-\varepsilon}\left(\sum_{j=0}^{\ell}\binom{\ell}{j}(-1)^j\left(\frac{2-j(1-y_{n+1})}{1+y_n}\right)_+^{n/2-1} f(\omega_x y_j)\right)\\ &\quad\cdot\frac{dy}{(1-y_{n+1})^{(n+\alpha)/2}}\end{aligned}$$

and an average kernel

$$(1.9)\qquad \lambda_{\ell,\alpha/2}(\eta)=\frac{\eta^{-1}}{\kappa_\ell(\alpha/2)\Gamma(1+\alpha/2)}\sum_{j=0}^{\ell}\binom{\ell}{j}(-1)^j(\eta-j)_+^{\alpha/2},\quad \ell>\alpha/2.$$

This kernel arises when inverting one-dimensional fractional integrals and has the following properties (see [14], [13]):

$$(1.10)\qquad \int_0^\infty \lambda_{\ell,\alpha/2}(\eta)d\eta=1,\qquad \lambda_{\ell,\alpha/2}(\eta)=\begin{cases}O(\eta^{\alpha/2-1}) & \text{if } \eta\in(0,1],\\ O(\eta^{\alpha/2-\ell-1}) & \text{if } \eta\in[1,\infty).\end{cases}$$

We introduce the analytical family of operators

$$(1.11)\qquad (M_t^\gamma\varphi)(x)=\sum_{m,\mu}\frac{m!\Gamma(n/2+\gamma)}{\Gamma(m+n/2+\gamma)}P_m^{(n/2+\gamma-1,n/2-\gamma-1)}(t)\varphi_{m,\mu}Y_{m,\mu}(x),$$

$$t\in[-1,1],\quad \mathrm{Re}\,\gamma>-\frac{n}{2},$$

being an approximate identity as $t\to 1$. If $\gamma=0$ the series (1.11) represents the mean value (1.2). In the case $\mathrm{Re}\gamma>0$ the operator M_t^γ is a spherical convolution

$$(1.12)\qquad (M_t^\gamma\nu)(x)=\int_{\Sigma_n}k_t^\gamma(xy)d\nu(y),$$

where $\nu\in M(\Sigma_n)$,

$$(1.13)\qquad k_t^\gamma(\tau)=\frac{\Gamma(n/2+\gamma)}{2\pi^{n/2}\Gamma(\gamma)}\frac{(\tau-t)_+^{\gamma-1}(1+\tau)^{1-n/2}}{(1-t)^{n/2+\gamma-1}}.$$

To prove the inversion formula (1.7) we need the following

LEMMA 1.1: *Let $f = I^\alpha \nu$, $\nu \in M(\Sigma_n)$, $0 < \alpha < n$. Then*

$$(1.14)\quad (T_\varepsilon^\alpha f)(x) = \int_0^\infty \lambda_{\ell,\alpha/2}(\eta)\Big(1-\frac{\varepsilon\eta}{2}\Big)_+^{(n-\alpha)/2-1}(M^{\alpha/2}_{1-\varepsilon\eta}\nu)(x)d\eta$$

$$(1.15)\quad = \int_{\Sigma_n} k_\varepsilon^{\ell,\alpha}(xy)d\nu(y),$$

where

$$(1.16)\quad k_\varepsilon^{\ell,\alpha}(\tau) = \frac{\Gamma(\frac{n+\alpha}{2})}{2\pi^{n/2}\Gamma(\alpha/2)}(1+\tau)^{1-n/2} \cdot \int_0^\infty \lambda_{\ell,\alpha/2}(\eta)\Big(1-\frac{\varepsilon\eta}{2}\Big)_+^{(n-\alpha)/2-1}(\tau-1+\varepsilon\eta)_+^{\alpha/2-1}(\varepsilon\eta)^{1-(n+\alpha)/2}d\eta.$$

Proof: Denote

$$h_{x,\nu}(t) = \frac{\Gamma(n/2)}{\Gamma(\frac{n+\alpha}{2})}(t+1)_+^{(n-\alpha)/2-1}(M_t^{\alpha/2}\nu)(x).$$

Let us prove the equality

$$(1.17)\quad g_{x,f}(\tau) = (I_+^{\alpha/2}h_{x,\nu})(\tau),$$

I_+^α being a fractional integral operator (5). It is sufficient to establish the equality of Fourier-Laplace coefficients of both sides of (1.17). By virtue of (1.4), (1.2), (7) we have

$$(g_{(\cdot),f}(\tau))_{m,\mu} = \frac{\Gamma(m+\frac{n-\alpha}{2})m!\Gamma(n/2)}{\Gamma(m+\frac{n+\alpha}{2})\Gamma(m+n/2)}P_m^{(n/2-1,n/2-1)}(\tau)(1+\tau)_+^{n/2-1}\nu_{m,\mu}.$$

The same expression may be obtained for Fourier-Laplace coefficients of the right-hand side if we use (1.11) and the formula 7.392(4) from [5]. Since the integral $(I_+^{\alpha/2}|h_{x,\nu}|)(1)$ is finite for almost all x then using (1.17), the equality

$$(T_\varepsilon^\alpha f)(x) = \frac{\Gamma(\frac{n+\alpha}{2})2^{1-(n-\alpha)/2}}{\Gamma(n/2)}(D_{+,\varepsilon}^{\alpha/2}g_{x,f})(1)$$

and the remark 2.1 from [13] we can obtain (1.14). The representation (1.15) may be derived from (1.14) by changing the order of integration. ∎

The integrand in (1.7) has a strong singularity at the point $\eta = 0$, therefore we treat the integral in (1.7) as $\lim_{\varepsilon\to 0}(T_\varepsilon^\alpha f)(x)$. In the general case $f = I^\alpha\nu$,

$\nu \in M(\Sigma_n)$, this limit will be understood in a weak sense. If $d\nu(y) = \varphi(y)dy$, $\varphi \in C(\Sigma_n)$, then it is natural to treat the $\lim_{\varepsilon\to 0}(T_\varepsilon^\alpha f)(x)$ in a uniform metrics. In the case $\varphi \in L_p(\Sigma_n)$ we use the a.e. convergence or the one in L_p-norm. As it is usual, the proof of the a.e. convergence is based on an estimate of the maximal operator $\varphi(x) \to \sup_{\varepsilon>0} |(T_\varepsilon^\alpha I^\alpha \varphi)(x)|$.

To prove such an estimate we obtain the general result for the maximal operator $(K^*\varphi)(x) = \sup_{\varepsilon>0} |(K_\varepsilon\varphi)(x)|$, where

$$(K_\varepsilon\varphi)(x) = \int_{\Sigma_n} k_\varepsilon(xy)\varphi(y)dy. \tag{1.18}$$

Denote $\sigma_t(x) = \{y \in \Sigma_n : xy > t\}$, where $t \in (-1,1)$, $x \in \Sigma_n$;

$$\varphi^*(x) = \sup_{t\in(-1,1)} \frac{1}{(1-t)^{n/2}} \int_{\sigma_t(x)} |\varphi(y)|dy$$
$$= \sup_{t\in(-1,1)} \frac{\sigma_{n-1}}{(1-t)^{n/2}} \int_t^1 (1-\tau^2)^{n/2-1}(M_\tau^0|\varphi|)(x)d\tau.$$

$$\varphi^{**}(x) = \sup_{t\in(-1,1)} \frac{1}{\text{mes}\,\sigma_t(x)} \int_{\sigma_t(x)} |\varphi(y)|dy$$

is a Hardy-Littlewood maximal function on Σ_n. It is easy to see that $c_1\varphi^*(x) \le \varphi^{**}(x) \le c_2\varphi^*(x)$ for some positive constants c_1, c_2 which depend only on n.

THEOREM 1.1: *Let*

$$|k_\varepsilon(1-\tau)| \le \frac{\tau^{1-n/2}}{\varepsilon}\lambda(\tau/\varepsilon), \tag{1.19}$$

$\lambda(\xi)$ being a non-increasing integrable function on $(0,\infty)$. Then

$$(K^*\varphi)(x) \le Ac_n\varphi^*(x), \qquad A = \int_0^\infty \lambda(\xi)d\xi,$$

c_n being a constant depending on n.

Proof: We may assume $\varphi \ge 0$. Using the argument of Theorem 2 from [17, p.64] we have

$$|(K_\varepsilon\varphi)(x) \le \sigma_{n-1}\int_0^{2/\varepsilon} \lambda(\xi)(2-\varepsilon\xi)^{n/2-1}(M_{1-\varepsilon\xi}^0\varphi)(x)d\xi \le A \sup_{0<h<2} \psi_{x,\varphi}(h),$$

where

$$\psi_{x,\varphi}(h) = \frac{\sigma_{n-1}}{h}\int_{1-h}^1 (1-\tau)^{1-n/2}[(1-\tau^2)^{n/2-1}(M_\tau^0\varphi)(x)]d\tau.$$

Let us estimate the last integral. We have

$$\psi_{x,\varphi}(h) = \frac{1}{h}\int_{1-h}^{1} u(\tau)dv(\tau),$$

where

$$u(\tau) = (1-\tau)^{1-n/2},$$

$$v(\tau) = -\sigma_{n-1} = \int_{\tau}^{1} (1-t^2)^{n/2-1}(M_t^0\varphi)(x)dt, \quad |v(\tau)| \le (1-\tau)^{n/2}\varphi^*(x).$$

Hence

$$\psi_{x,\varphi}(h) = \frac{1}{h}[uv|_{1-h}^{1} + (1-\frac{n}{2})\int_{1-h}^{1} v(\tau)(1-\tau)^{-n/2}d\tau$$

$$= -h^{-n/2}v(1-h) - \frac{n/2-1}{h}\int_{1-h}^{1} v(\tau)(1-\tau)^{-n/2}d\tau \le c(n)\varphi^*(x). \quad ∎$$

COROLLARY 1.1: *Let $\varphi \in L_p(\Sigma_n)$, $1 \le p \le \infty$. If $k_\varepsilon(xy)$ satisfies (1.19), then there exist constants c_1, c_2 depending only on n such that*

$$\|K^*\varphi\|_p \le c_1\|\varphi\|_p \quad if \;\; 1 < p \le \infty,$$

and

$$mes\,\{x \in \Sigma_n : (K^*\varphi)(x) > a\} \le \frac{c_2}{a}\|\varphi\|_1 \quad if \;\; p = 1, \; a > 0.$$

This assertion follows from the similar one for $\varphi^{**}(x)$. The latter may be verified using the scheme from [17] with insignificant variations when proving a covering lemma (these variations are caused by the compactness of Σ_n).

Definition 1.1: The approximate identity $\{A_\varepsilon\}_{\varepsilon\to+0}$ is called regular, if there exists $\delta > 0$ such that for all $\varepsilon \in (0,\delta)$ and for all $\varphi \in L_1(\Sigma_n)$ the function $(A_\varepsilon\varphi)(x)$ is represented by a spherical convolution (1.18) with a kernel $k_\varepsilon(xy)$ satisfying (1.19).

We have the following examples of regular approximate identities: a family (1.12) of operators M_t^γ with $\operatorname{Re}\gamma \ge 1$, $t = 1-\varepsilon$, $\delta = 2$; a family of Poisson operators $\varphi(x) \to \varphi(x,r)$, where $r = 1-\varepsilon$, $\delta = 1$. A family (1.11) with $\operatorname{Re}\gamma < 1$, $\varepsilon = 1-t$ is an example of a non-regular approximate identity.

THEOREM 1.2: *Let $f = I^\alpha\nu$, $0<\alpha<n$, $\nu \in M(\Sigma_n)$. Then*

$$\int_{\Sigma_n}\omega(x)d\nu(x) = \lim_{\varepsilon\to 0}\int_{\Sigma_n}\omega(x)(T_\varepsilon^\alpha f)(x)dx \tag{1.20}$$

for any $\omega \in L_\infty(\Sigma_n)$. In particular, the measure $\nu(\Omega)$, $\Omega \in \mathcal{B}(\Sigma_n)$, may be restored by the formula

$$\nu(\Omega) = \lim_{\varepsilon\to 0}\int_\Omega (T_\varepsilon^\alpha f)(x)dx. \tag{1.21}$$

If ν is an absolutely continuous measure (with respect to the Lebesgue measure on Σ_n) with the density $\varphi \in X_p(\Sigma_n)$, $1\le p\le\infty$, then

$$\varphi(x) = (T^\alpha f)(x) \equiv \mathit{lim}^{\text{a.e.}}_{\varepsilon\to 0}(T_\varepsilon^\alpha f)(x), \tag{1.22}$$

where the limit may be also treated in X_p-norm.

Proof: At first we consider the case $f = I^\alpha\varphi$, $\varphi\in X_p(\Sigma_n)$. Denote

$$(K_\varepsilon^{\ell,\alpha}\varphi)(x) = \int_{\Sigma_n} k_\varepsilon^{\ell,\alpha}(xy)\varphi(y)dy, \tag{1.23}$$

$k_\varepsilon^{\ell,\alpha}(\tau)$ being a kernel (1.16). Using the Funk-Hecke theorem [4, p.247] we have $(K_\varepsilon^{\ell,\alpha}Y_{m,\mu})(x) = k_{\varepsilon,m}^{\ell,\alpha}Y_{m,\mu}(x)$, where according to (1.14), (1.11) the multiplier $k_{\varepsilon,m}^{\ell,\alpha}$ has the form

$$k_{\varepsilon,m}^{\ell,\alpha} = \frac{m!\Gamma(\frac{n\pm\alpha}{2})}{\Gamma(m+\frac{n\pm\alpha}{2})} \cdot\int_0^\infty \lambda_{\ell,\alpha/2}(\eta)\Big(1-\frac{\varepsilon\eta}{2}\Big)_+^{(n-\alpha)/2-1} P_m^{((n+\alpha)/2-1,(n-\alpha)/2-1)}(1-\varepsilon\eta)d\eta.$$

By virtue of (1.10) $\lim_{\varepsilon\to 0}k_{\varepsilon,m}^{\ell,\alpha} = 1$. Thus, the relation

$$\lim_{\varepsilon\to 0}(K_\varepsilon^{\ell,\alpha}\varphi)(x) = \varphi(x) \tag{1.24}$$

holds on the set $\mathcal{Y}(\Sigma_n)$ which is dense in $X_p(\Sigma_n)$. In order to extend this relation to functions $\varphi\in X_p(\Sigma_n)$ it is sufficient to prove the regularity of the approximative identity $\{K_\varepsilon^{\ell,\alpha}\}$. Indeed, if (1.19) holds for $k_\varepsilon^{\ell,\alpha}$, then we have the following uniform estimate

(1.25)
$$\|K_\varepsilon^{\ell,\alpha}\|_{X_p} \le \frac{\sigma_{n-1}}{\varepsilon}\|\varphi\|_{X_p}\int_{-1}^1 \lambda\Big(\frac{1-\tau}{\varepsilon}\Big)(1+\tau)^{n/2-1}d\tau \le A\tilde{c}\|\varphi\|_{X_p},\ \tilde{c}=\tilde{c}(n),$$

that leads to the equality

$$\lim_{\varepsilon\to 0}\|K_\varepsilon^{\ell,\alpha}\varphi-\varphi\|_{X_p}=0. \tag{1.26}$$

This equality in conjunction with the convergence $K_\varepsilon^{\ell,\alpha}Y_{m,\mu}\to Y_{m,\mu}$ and with the Corrollary 1.1 provides the convergence $(K_\varepsilon^{\ell,\alpha}\varphi)(x)\to\varphi(x)$ almost everywhere. The validity of (1.19) for $k_\varepsilon^{\ell,\alpha}(\tau)$ follows from the estimate

$$|k_\varepsilon^{\ell,\alpha}(1-\tau)|\le c(n)\frac{\tau^{1-n/2}}{\varepsilon}\begin{cases}(\tau/\varepsilon)^{\alpha/2-1} & \text{if } \tau<\varepsilon,\\ (\tau/\varepsilon)^{\alpha/2-\ell-1} & \text{if } \tau>\varepsilon,\end{cases}$$

that holds for $\varepsilon\le 1$ and may be verified easily.

Now let $f=I^\alpha\nu$, $\nu\in M(\Sigma_n)$. According to Lemma 1.1 for any $\omega\in L_\infty(\Sigma_n)$ we have

$$\int_{\Sigma_n}\omega(x)(T_\varepsilon^\alpha f)(x)dx=\int_{\Sigma_n}(K_\varepsilon^{\ell,\alpha}\omega)(y)d\nu(y)\to\int_{\Sigma_n}\omega(y)d\nu(y)$$

as $\varepsilon\to 0$. The passage to the limit is true due to Lebesgue dominated convergence theorem with regard to relations:

$$|(K_\varepsilon^{\ell,\alpha}\omega)(y)|\le A\tilde{c}\|\omega\|_\infty,\qquad \lim\nolimits^{\text{a.e.}}_{\varepsilon\to 0}(K_\varepsilon^{\ell,\alpha}\omega)(y)=\omega(y). \qquad \blacksquare$$

2. The inversion of spherical potentials with a logarithmic kernel

Let us consider the following integral operator:

$$(I_\gamma^n\nu)(x)=\frac{2^{1-n}}{\pi^{n/2}\Gamma(n/2)}\int_{\Sigma_n}\log\frac{\gamma}{|x-y|}d\nu(y), \tag{2.1}$$

assuming γ to be a fixed positive number. The Riesz potential $I^\alpha\varphi$ of the order $\alpha=n$ may be defined as the operator (2.1). Really, it is not hard to show that

$$I_\alpha^n\nu=\lim_{\alpha\to n}\left(I^\alpha\nu-c_{n,\alpha}\gamma^{\alpha-n}\int_{\Sigma_n}d\nu(x)\right)$$

and $(I_\gamma^n\nu)_{m,\mu}=k_m^n\nu_{m,\mu}$, where $k_m^n=\dfrac{\Gamma(m)}{\Gamma(m+n)}$ if $m\ge 1$ and

$$k_0^n=\frac{1}{\Gamma(n)}\left[2\log\frac{\gamma}{2}+\psi(n)-\psi(n/2)\right],\quad \psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}.$$

THEOREM 2.1: *Let $f = I^n_\gamma \nu$, $k^n_0 \neq 0$, $\nu \in M(\Sigma_n)$, $T^n_\varepsilon f$ be a truncated hypersingular integral of the form (1.8). Then*

$$\int_{\Sigma_n} \omega(x)d\nu(x) = \frac{1}{\sigma_n k^n_0}\int_{\Sigma_n}\omega(x)dx\int_{\Sigma_n} f(x)dx + \lim_{\varepsilon\to 0}\int_{\Sigma_n}\omega(x)(T^n_\varepsilon f)(x)dx$$

for any $\omega \in L_\infty(\Sigma_n)$. In particular,

$$\nu(\Omega) = \frac{\text{mes}\,\Omega}{\sigma_n k^n_0}\int_{\Sigma_n} f(x)dx + \lim_{\varepsilon\to 0}\int_\Omega (T^n_\varepsilon f)(x)dx, \tag{2.2}$$

mes Ω being a Lebesque measure of Ω. If ν is an absolutly continuous measure with the density $\varphi \in X_p(\Sigma_n)$, then

$$\varphi(x) = \frac{1}{\sigma_n k^n_0}\int_{\Sigma_n} f(x)dx + (T^n f)(x), \tag{2.3}$$

where $(T^n f)(x) = \lim^{\text{a.e.}}_{\varepsilon\to 0}(T^n_\varepsilon f)(x) = \lim^{(X_p)}_{\varepsilon\to 0}(T^n_\varepsilon)(x)$.

Proof: If $0 < \alpha < n$, $\ell > n/2$, then (1.14)-(1.16) yield

$$\begin{aligned} T^\alpha_\varepsilon[I^\alpha\nu - c_{n,\alpha}\gamma^{\alpha-n}\nu(\Sigma_n)] &= \int_{\Sigma_n}\tilde{k}^{\ell,\alpha}_\varepsilon(xy)d\nu(y) \\ &+ \mu_\alpha\nu(\Sigma_n)\int_0^\infty \lambda_{\ell,\alpha/2}(\eta)\Big(1-\frac{\varepsilon\eta}{2}\Big)_+^{(n-\alpha)/2-1}d\eta, \end{aligned} \tag{2.4}$$

where

$$\tilde{k}^{\ell,\alpha}_\varepsilon(\tau) = \int_0^\infty \lambda_{\ell,\alpha/2}(\eta)\Big(1-\frac{\varepsilon\eta}{2}\Big)_+^{(n-\alpha)/2-1}$$
$$\times\left[\frac{\Gamma(\frac{n+\alpha}{2})(1+\tau)^{1-n/2}(\tau-1+\varepsilon\eta)_+^{\alpha/2-1}}{2\pi^{n/2}\Gamma(\alpha/2)(\varepsilon\eta)^{(n+\alpha)/2-1}} - \frac{1}{\sigma_n}\right]d\eta,$$

$$\mu_\alpha = 2^{-(\alpha+n)/2}\gamma^{\alpha-n}\Gamma\Big(\frac{n+\alpha}{2}\Big)/\pi^{n/2}\Gamma(\alpha/2)\sigma_n.$$

We note, that

$$\lim_{\alpha\to n}\mu_\alpha\int_0^\infty \lambda_{\ell,\alpha/2}(\eta)\Big(1-\frac{\varepsilon\eta}{2}\Big)_+^{(n-\alpha)/2-1}d\eta = \frac{\mu_*}{\varepsilon}\lambda_{\ell,n/2}\Big(\frac{2}{\varepsilon}\Big), \tag{2.5}$$

where $\mu_* = 4\lim_{\alpha\to n}\mu_\alpha/(n-\alpha)$. Really, decomposing the integral in the left-hand side into two integrals (from zero to $1/\varepsilon$ and from $1/\varepsilon$ to $2/\varepsilon$) and using

(1.10) we can prove that the first term tends to zero and for the second one the following relation holds:

$$\mu_\alpha \int_{1/\varepsilon}^{2/\varepsilon} \lambda_{\ell,\alpha/2}(\eta)\Big(1-\frac{\varepsilon\eta}{2}\Big)^{(n-\alpha)/2-1} d\eta$$

$$= \frac{2^{(\alpha-n)/2+2}\mu_\alpha}{(n-\alpha)\varepsilon}\lambda_{\ell,\alpha/2}\Big(\frac{1}{\varepsilon}\Big) + \frac{4\mu_\alpha}{(n-\alpha)\varepsilon}\int_{1/\varepsilon}^{2/\varepsilon} \lambda'_{\ell,\alpha/2}(\eta)\Big(1-\frac{\varepsilon\eta}{2}\Big)^{n-\alpha/2} d\eta$$

$$\to \frac{\mu_*}{\varepsilon}\lambda_{\ell,n/2}\Big(\frac{2}{\varepsilon}\Big), \quad \alpha \to n.$$

If we pass to the limit in (2.4) as $\alpha \to n$, then by virtue of (2.5) we have

$$T_\varepsilon^n I_\gamma^n \nu = \int_{\Sigma_n} \tilde{k}_\varepsilon^{\ell,n}(xy)d\nu(y) + \nu(\Sigma_n)\frac{\mu^*}{\varepsilon}\lambda_{\ell,n/2}\Big(\frac{2}{\varepsilon}\Big). \tag{2.6}$$

Assume $d\nu(y) = \varphi(y)dy$, $\varphi \in X_p(\Sigma_n)$ and represent the kernel $\tilde{k}_\varepsilon^{\ell,n}(\tau)$ in the form

$$\tilde{k}_\varepsilon^{\ell,n}(\tau) = k_{1,\varepsilon}(\tau) + k_{2,\varepsilon} + k_{3,\varepsilon}(\tau), \tag{2.7}$$

where

$$k_{1,\varepsilon}(\tau) = \frac{\Gamma(n)\pi^{-n/2}}{2\Gamma(n/2)}(1+\tau)^{1-n/2} \cdot \int_0^{1/\varepsilon} \lambda_{\ell,n/2}(\eta)\Big(1-\frac{\varepsilon\eta}{2}\Big)^{-1}\frac{(\tau-1+\varepsilon\eta)_+^{n/2-1}}{(\varepsilon\eta)^{n-1}}d\eta,$$

$$k_{2,\varepsilon} = \frac{1}{\sigma_n}\int_0^{1/\varepsilon} \lambda_{\ell,n/2}(\eta)\Big(1-\frac{\varepsilon\eta}{2}\Big)^{-1} d\eta,$$

$$k_{3,\varepsilon}(\tau) = \frac{\Gamma(n)\pi^{-n/2}}{2\Gamma(n/2)}\int_{1/\varepsilon}^{2/\varepsilon} \lambda_{\ell,n/2}(\eta)\Big(1-\frac{\varepsilon\eta}{2}\Big)^{-1} \cdot \left[\frac{(1+\tau)^{1-n/2}(\tau-1+\varepsilon\eta)_+^{n/2-1}}{(\varepsilon\eta)^{n-1}} - 2^{1-n}\right] d\eta.$$

Let

(2.8)

$$(K_{1,\varepsilon}\varphi)(x) = \int_{\Sigma_n} k_{1,\varepsilon}(xy)\varphi(y)dy = \int_0^{1/\varepsilon} \lambda_{\ell,n/2}(\eta)\Big(1-\frac{\varepsilon\eta}{2}\Big)^{-1}(M_{1-\varepsilon\eta}^{n/2}\varphi)(x)d\eta.$$

As in (1.22) $\lim^{\text{a.e.}}_{\varepsilon\to 0}(K_{1,\varepsilon}\varphi)(x) = \lim^{(X_p)}_{\varepsilon\to 0}(K_{1,\varepsilon}\varphi)(x) = \varphi(x)$. By virtue of (1.10) we have $\lim_{\varepsilon\to 0} k_{2,\varepsilon} = 1/\sigma_n$. The kernel $k_{3,\varepsilon}(\tau)$ admits the estimate

$$(2.9) \qquad |k_{3,\varepsilon}(\tau)| \le C\varepsilon^{\ell-n/2} h(\tau), \quad h(\tau) = \begin{cases} 1, & \tau > 0, \\ 1 + \log \frac{1}{1+\tau}, & \tau < 0, \end{cases}$$

that yields the inequality

$$\left| \int_{\Sigma_n} k_{3,\varepsilon}(xy)\varphi(y)dy \right| \le C\varepsilon^{\ell-n/2} \int_{\Sigma_n} h(xy)|\varphi(y)|dy.$$

The relation (2.6) and the argument above leads to the following equalities

$$(2.10) \qquad \lim^{\text{a.e.}}_{\varepsilon\to 0}(T^n_\varepsilon I^n_\gamma\varphi)(x) = \lim^{(X_p)}_{\varepsilon\to 0}(T^n_\varepsilon I^n_\gamma\varphi)(x) = \varphi(x) - \frac{1}{\sigma_n}\int_{\Sigma_n}\varphi(x)dx,$$

that give (2.3). Let us consider the general case $f = I^n_\gamma \nu$, $\nu \in M(\Sigma_n)$. Given an arbitrary $\omega \in L_\infty(\Sigma_n)$, according to (2.6) we have

$$\lim_{\varepsilon\to 0}\int_{\Sigma_n}\omega(x)(T^n_\varepsilon f)(x)dx = \lim_{\varepsilon\to 0}\int_{\Sigma_n}(K_{1,\varepsilon}\omega)(y)d\nu(y) - \frac{\nu(\Sigma_n)}{\sigma_n}\int_{\Sigma_n}\omega(x)dx$$

$$= \int_{\Sigma_n}\omega(y)d\nu(y) - \frac{1}{\sigma_n k^n_0}\int_{\Sigma_n}\omega(x)dx\int_{\Sigma_n}f(x)dx. \qquad \blacksquare$$

Remark 2.1: The inequality $k^n_0 \neq 0$ holds for any $\gamma \ge 2$. If $0 < \gamma < 2$, then $k^n_0 \neq 0$ in the following cases:

1) n is even and $\log\frac{\gamma}{2}$ is irrational;
2) n is odd and $\log\gamma$ is irrational.

In another cases the equality $k^n_0 = 0$ may be true (e.g., $n = 1$ and $\gamma = 1$, or $n = 2$ and $\gamma = 2/\sqrt{e}$). We investigate these critical cases in Section 5.

3. The inversion of Riesz potentials by means of hypersingular operators containing a Poisson integral

The direct regularization method used in previous sections may also be applied for Riesz potentials of the orders $\alpha > n$. But the consideration of such α's in the frame of this method is connected with cumbersome technicalities, so we prefer to exhibit another approach which is based on the representation of $I^\alpha\varphi$ via the Poisson integral and covers all positive α.

Denote

$$(\mathcal{I}^{n,\alpha}\psi)(r) = \frac{r^{1-(n+\alpha)/2}}{\Gamma(\alpha)}\int_0^r \psi(\rho)\rho^{(n-\alpha)/2-1}(r-\rho)^{\alpha-1}d\rho, \quad 0 < \alpha < n.$$

LEMMA 3.1: *If $0<\alpha<n$, $\varphi\in L_1(\Sigma_n)$, then*

$$(I^\alpha\varphi)(x,r)=\Big(\mathcal{I}^{n,\alpha}\varphi(x,\cdot)\Big)(r). \tag{3.1}$$

In particular,

$$(I^\alpha\varphi)(x)=\frac{1}{\Gamma(\alpha)}\int_0^1\rho^{(n-\alpha)/2-1}(1-\rho)^{\alpha-1}\varphi(x,\rho)d\rho. \tag{3.2}$$

Proof: Changing the order of integration we obtain

$$[(\mathcal{I}^{n,\alpha}\varphi(x,\cdot))(r)]_{m,\mu}=\varphi_{m,\mu}(\mathcal{I}^{n,\alpha}\rho^m)(r)=r^m\frac{\Gamma(m+\frac{n-\alpha}{2})}{\Gamma(m+\frac{n+\alpha}{2})}\varphi_{m,\mu},\quad r\in(0,1],$$

that gives (3.1), (3.2). ∎

Using (3.1) we may solve the equation $I^\alpha\varphi=f$ by the following way. Let us apply the Poisson operator $P_r: f(x)\to f(x,r)$ to both sides of $I^\alpha\varphi=f$ and rewrite the result in the form

$$\frac{1}{\Gamma(\alpha)}\int_0^r(r-\rho)^{\alpha-1}\rho^{(n-\alpha)/2-1}\varphi(x,\rho)d\rho=r^{(n+\alpha)/2-1}f(x,r).$$

If we invert the fractional integral operator in the left-hand side by means of Marchaud's derivative (see [14], [8]) and then set $r=1$, we obtain the following formula

$$\varphi(x)=\frac{1}{\kappa_\ell(\alpha)}\int_0^\infty\eta^{-\alpha-1}\left[\sum_{j=0}^{\ell}\binom{\ell}{j}(-1)^j(1-j\eta)_+^{(n+\alpha)/2-1}f(x,1-j\eta)\right]d\eta$$

$$\stackrel{\text{def}}{=}(\mathcal{T}^\alpha f)(x). \tag{3.3}$$

Let us give a strict proof of this formula. Define $I^\alpha\nu$ for all $\alpha>0$, $\nu\in M(\Sigma_n)$ assuming

$$(I^\alpha\nu)(x)\sim\sum_{m,\mu}k_m^\alpha\nu_{m,\mu}Y_{m,\mu}(x), \tag{3.4}$$

where

$$k_m^\alpha=\begin{cases}\dfrac{\Gamma(m+\frac{n-\alpha}{2})}{\Gamma(m+\frac{n+\alpha}{2})} & \text{if}\quad \dfrac{\alpha-n}{2}\notin\mathbb{Z}_+,\ m\geq 0\\ & \text{and if}\quad (\alpha-n)/2=k\in\mathbb{Z}_+,\ m>k;\\ c_m & \text{if}\quad (\alpha-n)/2=k\in\mathbb{Z}_+,\ m\leq k,\end{cases}$$

$\{c_m\}$ being an arbitrary sequence of complex numbers different from zero.

The operator (3.4) is bounded from $M(\Sigma_n)$ into $L_1(\Sigma_n)$ (it follows, e.g., from Lemma 4.1 below). Consider the inversion problem for the operator (3.4). We denote

(3.5)
$$(\mathcal{T}_\varepsilon^\alpha f)(x) = \frac{1}{\kappa_\ell(\alpha)} \int_\varepsilon^1 \eta^{-\alpha-1} \left[\sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j (1-j\eta)_+^{(n+\alpha)/2-1} f(x, 1-j\eta) \right] d\eta,$$

$$\ell > \alpha$$

LEMMA 3.2: *Let $\alpha > 0$, $1 \le p \le \infty$. Then*

$$\lim\nolimits_{\varepsilon\to 0}^{(X_p)} (\mathcal{T}_\varepsilon^\alpha, Y_{m,\mu})(x) = \frac{\Gamma(m+\frac{n+\alpha}{2})}{\Gamma(m+\frac{n-\alpha}{2})} Y_{m,\mu}(x). \tag{3.6}$$

Proof: Let us continue the following obvious equality

$$\frac{1}{\Gamma(\lambda)} \int_0^1 \eta^{\lambda-1} (1-\eta)^{(n-\lambda)/2+m-1} d\eta = \frac{\Gamma(m+\frac{n-\lambda}{2})}{\Gamma(m+\frac{n+\lambda}{2})}, \qquad 0 < \operatorname{Re}\lambda < 1,$$

analytically to the strip $-\ell < \operatorname{Re}\lambda < 0$, $\ell \in \mathbb{N}$. Representing the analytical continuation of the left-hand side in a difference integral form (see [13]) we have

$$\frac{1}{\kappa_\ell(-\lambda)} \int_0^1 \eta^{\lambda-1} \left(\sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j (1-j\eta)^{(n-\lambda)/2+m-1} \right) d\eta = \frac{\Gamma(m+\frac{n-\lambda}{2})}{\Gamma(m+\frac{n+\lambda}{2})}.$$

Hence

(3.7)
$$\frac{1}{\kappa_\ell(\alpha)} \int_0^1 \eta^{-\alpha-1} \left(\sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j (1-j\eta)^{(n+\alpha)/2+m-1} \right) d\eta = \frac{\Gamma(m+\frac{n+\alpha}{2})}{\Gamma(m+\frac{n-\alpha}{2})}$$

for $\alpha \in (0,\ell)$. It is easy to see that

$$(\mathcal{T}_\varepsilon^\alpha Y_{m,\mu})(x) = a_m^{\ell,\alpha}(\varepsilon) Y_{m,\mu}(x), \tag{3.8}$$

where

$$a_m^{\ell,\alpha}(\varepsilon) = \frac{1}{\kappa_\ell(\alpha)} \int_\varepsilon^1 \eta^{-\alpha-1} \left(\sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j (1-j\eta)^{(n+\alpha)/2+m-1} \right) d\eta.$$

The equality (3.6) follows from (3.7) and (3.8). ∎

THEOREM 3.1: *Let $f = I^\alpha \nu$ be the potential (3.4), $\alpha > 0$, $\nu \in M(\Sigma_n)$. Then the limit*

$$(f^{(\alpha)}, \omega) \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0} \int_{\Sigma_n} \omega(x)(\tau_\varepsilon^\alpha f)(x)dx$$

exists for any $\omega \in L_\infty(\Sigma_n)$, and

$$(3.9) \qquad (\nu, \omega) = \begin{cases} (f^{(\alpha)}, \omega) & if \ (\alpha - n)/2 \in \mathbb{Z}_+, \\ (f^{(\alpha)}, \omega) + \sum_{m=0}^{\kappa} \sum_{\mu} \frac{f_{m,\mu}\omega_{m,\mu}}{c_m} & if \ (\alpha - n)/2 = k \in \mathbb{Z}_+. \end{cases}$$

In particular,

$$(3.10) \qquad \nu(\Omega) = \begin{cases} \lim_{\varepsilon \to 0} \int_\Omega (\mathcal{T}_\varepsilon^\alpha f)(x)dx \quad if \quad (\alpha - n)/2 \in \mathbb{Z}_+, \\ \lim_{\varepsilon \to 0} \int_\Omega (\mathcal{T}_\varepsilon^\alpha f)(x)dx + \sum_{m=0}^{k} \sum_{\mu} \frac{f_{m,\mu}}{c_m} \int_\Omega Y_{m,\mu}(x)dx \\ if \quad (\alpha - n)/2 = k \in \mathbb{Z}_+. \end{cases}$$

If ν is an absolutely continuous measure with the density $\varphi \in X_p(\Sigma_n)$, then

1) *there exists the limit $(\mathcal{T}^\alpha f)(x) = \lim^{\text{a.e.}}_{\varepsilon \to 0} (\mathcal{T}_\varepsilon^\alpha f)(x)$, treated also in X_p-norm;*
2) *the following inversion formula holds:*

$$(3.11) \qquad \varphi(x) = \begin{cases} (\mathcal{T}^\alpha f)(x) & if \quad (\alpha - n)/2 \notin \mathbb{Z}_+, \\ (\mathcal{T}^\alpha f)(x) + \sum_{m=0}^{k} \sum_{\mu} \frac{f_{m,\mu}}{c_m} Y_{m,\mu}(x) & if \quad (\alpha - n)/2 = k \in \mathbb{Z}_+. \end{cases}$$

Proof: Let us fix $s(\in \mathbb{Z}_+) > (\alpha - n)/2 - 1$, and assume

$$(A_s\varphi)(x) = \varphi(x) - \sum_{m=0}^{s} \sum_{\mu} \varphi_{m,\mu} Y_{m,\mu}(x), \qquad \varphi \in L_1(\Sigma_n).$$

Given $\nu \in M(\Sigma_n)$, we denote

$$(A_s\nu)(\Omega) = \nu(\Omega) - \sum_{m=0}^{s} \sum_{\mu} \nu_{m,\mu} \int_\Omega Y_{m,\mu}(x)dx, \qquad \Omega \in \mathcal{B}(\Sigma_n).$$

It is not hard to show that

$$(3.12) \qquad |(A_s Ym, \mu)(x, \rho)| \le \rho^{s+1} |Ym, \mu(x)| \quad \forall Ym, \mu(x) \in \mathcal{Y}(\Sigma_n)$$

and

$$I_+^\alpha[\rho_+^{(n-\alpha)/2-1}(A_S Y_{m,\mu})(x,\rho)](r) = r^{(n+\alpha)/2-1}(A_S I^\alpha Y_{m,\mu})(x,r). \tag{3.13}$$

Let us extent (3.13) to all measures $\nu \in M(\Sigma_n)$. Since

$$\left|\int_{\Sigma_n} \omega(x)(A_S\nu)(x,\rho)dx\right| = \left|\int_{\Sigma_n} (A_S\omega)(x,\rho)d\nu(x)\right|$$
$$\le \|(A_S\omega)(\cdot,\rho)\|_C\|\nu\|_M \le \rho^{s+1}\|\nu\|_M\|\omega\|_C$$

for any $\omega \in C(\Sigma_n)$, then

$$\|(A_S\nu)(\cdot,\rho)\|_{L_1(\Sigma_n)} \le \rho^{s+1}\|\nu\|_{M(\Sigma_n)}, \tag{3.14}$$

and therefore $I_+^\alpha[\rho^{(n-\alpha)/2-1}(A_S\nu)(x,\rho)](r) \in L_1(\Sigma_n)$.

Now we can assert the equality

$$I_+^\alpha[\rho^{(n-\alpha)/2-1}(A_S\nu)(x,\rho)](r) = r^{(n+\alpha)/2-1}(A_S I^\alpha\nu)(x,r) \tag{3.15}$$

to be valid since the Fourier-Laplace coefficients of its both sides coincide by virtue of (3.13). Using for $f = I^\alpha\nu$ the known scheme of inverting of fractional integrals (see [14], [13]) we have

$$\begin{aligned}\mathcal{D}_{+,\varepsilon}^\alpha&[\rho^{(n+\alpha)/2-1}(A_S f)(x,\rho)](r)\\ &= \int_0^\infty \lambda_{\ell,\alpha}(\eta)(\eta-\varepsilon\eta)_+^{(n-\alpha)/2-1}(A_S\nu)(x,r-\varepsilon\eta)d\eta,\end{aligned} \tag{3.16}$$

$\lambda_{\ell,\alpha}$ being a kernel of the form (1.9).

Denote

$$(\Lambda_\varepsilon\nu)(x) = \int_0^\infty \lambda_{\ell,\alpha}(\eta)(1-\varepsilon\eta)_+^{e(n-\alpha)/2-1}\nu(x,1-\varepsilon\eta)d\eta.$$

If ν is an absolutely continuous measure with a density φ, we shall write $\Lambda_\varepsilon\varphi$ instead of $\Lambda_\varepsilon\nu$. Let us rewrite (3.16) in the form $(\mathcal{T}_\varepsilon^\alpha A_s f)(x,r) = (\Lambda_\varepsilon A_s\nu)(x,r)$ and go to the limit as $r \to 1$. We obtain

$$(\mathcal{T}_\varepsilon^\alpha A_s f)(x) = (\Lambda_\varepsilon A_s\nu)(x). \tag{3.17}$$

Suppose ν to be an absolutely continuous measure with a density $\varphi \in X_p(\Sigma_n)$. If we prove that

$$\lim_{\varepsilon\to 0}^{\text{a.e.}}(\mathcal{T}_\varepsilon^\alpha A_s f)(x) = (A_s\varphi)(x)$$

then, using the equality

$$(3.18)\qquad (\mathcal{T}_\varepsilon^\alpha A_s f)(x) = (\mathcal{T}_\varepsilon^\alpha f)(x) - \sum_{m=0}^{s} \sum_{\mu} f_{m,\mu} (\mathcal{T}_\varepsilon^\alpha Y_{m,\mu})(x)$$

and Lemma 3.2, we obtain the a.e. convergence of the integral $(\tau^\alpha f)(x)$ and the formula

$$(3.19)\qquad A_s\varphi = \lim\nolimits^{\text{a.e.}}_{\varepsilon\to 0} \mathcal{T}_\varepsilon^\alpha f - \sum_{m=0}^{s} \sum_{\mu} f_{m,\mu} \frac{\Gamma(m + \frac{n+\alpha}{2})}{\Gamma(m + \frac{n-\alpha}{2})} Y_{m,\mu}$$

that gives (3.11). Let

$$\begin{aligned}(\Lambda_\varepsilon A_s\varphi)(x) &= \Big(\int_0^{1/2\varepsilon} + \int_{1/2\varepsilon}^{1/\varepsilon} \lambda_{\ell,\alpha}(\eta)(1-\varepsilon\eta)^{(n-\alpha)/2-1}(A_s\varphi)(x, 1-\varepsilon\eta)d\eta\Big)\\ (3.20)\qquad &= \Lambda_{\varepsilon,1}\varphi + \Lambda_{\varepsilon,2}\varphi.\end{aligned}$$

(if $(\alpha - n)/2 = k \in \mathbb{Z}_+$ we assume $s = k$). By virtue of (1.10) and according to relations

$$\sup_{0<r<1} |(A_s\varphi)(x,r)| \le C(A_s,\varphi)^*(x), \qquad \lim\nolimits^{\text{a.e.}}_{r\to 1} (A_s\varphi)(x,r) = (A_s\varphi)(x)$$

the first integral tends to $(A_s\varphi)(x)$. The second one tends to zero since

$$\begin{aligned}|\Lambda_{\varepsilon,2}\varphi| &\le C \int_{1/2\varepsilon}^{1/\varepsilon} \eta^{\alpha-\ell-1}(1-\varepsilon\eta)^{(n-\alpha)/2-1}|(A_S\varphi)(x, 1-\varepsilon\eta)|d\eta\\ &= \varepsilon^{\ell-\alpha} \int_0^{1/2} \rho^{(n-\alpha)/2-1}|(A_S\varphi)(x,\rho)|(1-\rho)^{\alpha-\ell-1}d\rho.\end{aligned}$$

Using (3.20) and the relations

$$\sup_{0<r<1} \|(A_S\varphi)(\cdot,r)\|_{X_p} \le \|A_S\varphi\|_{X_p}, \quad \lim\nolimits^{(X_p)}_{r\to 1} (A_S\varphi)(x,r) = (A_S\varphi)(x),$$

it is easy to show that $\lim_{\varepsilon\to 0} \|\mathcal{T}_\varepsilon^\alpha A_S f - A_S\varphi\|_{X_p} = 0$.

The last equality leads to the formula (3.9), in which the hypersingular integral $(\mathcal{T}^\alpha f)(x)$ is treated as a limit in X_p-norm. If $f = I^\alpha\nu$, $\nu \in M(\Sigma_n)$, then by virtue of (3.5), (3.17) for any $\omega \in L_\infty(\Sigma_n)$ we have

$$\lim_{\varepsilon\to 0}\int_{\Sigma_n}\omega(x)(\mathcal{T}_\varepsilon^\alpha f)(x)dx-\sum_{m=0}^{s}\sum_{\mu}\frac{\Gamma(m+\frac{n+\alpha}{2})}{\Gamma(m+\frac{n-\alpha}{2})}f_{m,\mu}\omega_{m,\mu}$$

$$=\lim_{\varepsilon\to 0}\int_{\Sigma_n}\omega(x)(\mathcal{T}_\varepsilon^\alpha A_s f)(x)dx=\lim_{\varepsilon\to 0}\int_{\Sigma_n}\omega(x)(\Lambda_\varepsilon A_s\nu)(x)dx$$

$$=\lim_{\varepsilon\to 0}\int_{\Sigma_n}(\Lambda_\varepsilon\omega)(y)d(A_s\nu)(y)=\int_{\Sigma_n}\omega(y)d(A_s\nu)(y)$$

$$=(\nu,\omega)-\sum_{m=0}^{s}\sum_{\mu}\nu_{m,\mu}\omega_{m,\mu},$$

that gives (3.7), (3.8). ∎

4. The description of spaces $L_p^\alpha(\Sigma_n)$, $C^\alpha(\Sigma_n)$, $M^\alpha(\Sigma_n)$

It is convenient to use the unique notation $X(\Sigma_n)$ for spaces $L_p(\Sigma_n)$ ($1 \le p \le \infty$), $C(\Sigma_n)$, $M(\Sigma_n)$ and the notation $X^\alpha(\Sigma_n)$ for corresponding spaces $L_p^\alpha(\Sigma_n)$, $C^\alpha(\Sigma_n)$, $M^\alpha(\Sigma_n)$. We denote by $\mathring{X}(\Sigma_n)$ a subspace of the space $X(\Sigma_n)$ that consists of functions (or measures) with a zero mean value. Let us redenote the operator (3.4) by I_0^α and consider the following spaces generated by fractional integrals (3.4), (8)-(11):

$$I_j^\alpha(X)=\{f: f=I_j^\alpha\varphi,\ \varphi\in X(\Sigma_n)\},\ j=0,1,2, \tag{4.1}$$

$$I_j^\alpha(\mathring{X})=\{f: f=I_j^\alpha\varphi,\ \varphi\in \mathring{X}(\Sigma_n)\},\ j=3,4, \tag{4.2}$$

with norms defined as the corresponding norms of φ. The spaces (4.2) do not contain constants, therefore we also introduce the spaces

$$\mathbb{C}+I_j^\alpha(\mathring{X})=\{f: f=c+I_j^\alpha\varphi,\ \ c\in\mathbb{C},\ \ \varphi\in\mathring{X}(\Sigma_n)\},\ \ j=3,4, \tag{4.3}$$

with the norms

$$\|f\|_{\mathbb{C}+I_j^\alpha(\mathring{X})}=\|c+I_j^\alpha\varphi\|_{\mathbb{C}+X_j^\alpha(\mathring{X})}\overset{\text{def}}{=}|c|+\|\varphi\|_{\mathring{X}(\Sigma_n)}.$$

We need the following auxiliary assertion.

LEMMA 4.1: *If the multiplier* $\{k_m\}_{m=0}^{\infty}$ *of the operator* K *satisfies the asymptotic relation*

$$k_m = \sum_{j=0}^{N-1} \frac{c_j}{m^{\lambda+j}} + O(m^{-\lambda-N}), \quad m \to \infty, \tag{4.4}$$

where $\lambda \geq 0$, $\lambda + N > n$, *then* K *is a bounded operator in* $X(\Sigma_n)$.

Proof: We rewrite (4.4) in the form

$$k_m = \sum_{j=0}^{N-1} \frac{\tilde{c}_j}{(m+1)^{\lambda+j}} + \tilde{k}_m, \qquad \tilde{k}_m = O(m^{-\lambda-N}), \qquad m \to \infty.$$

According to Lemma 1 from [15] the operator $\tilde{K}$ generated by $\{\tilde{\kappa}_m\}$ is a spherical convolution with a continuous function defined on $[-1,1]$. The operators (9), corresponding to multipliers $\{(m+1)^{-\lambda-j}\}$, are bounded in spaces under consideration. This gives the required result. ∎

LEMMA 4.2: *The spaces* $X^\alpha(\Sigma_n)$, $I_j^\alpha(X)$ $(j = 0,1,2)$, $\mathbb{C} + I_j^\alpha(\overset{\circ}{X})$ $(j = 3,4)$ *coinside up to the equivalence of the norms.*

Proof: The relation $X^\alpha(\Sigma_n) = I_2^\alpha(X)$ follows from the definition of $X^\alpha(\Sigma_n)$. The relations $I_2^\alpha(X) = I_0^\alpha(X) = I_1^\alpha(X)$ follow by virtue of Lemma 4.1 from the equality

$$\frac{1}{(m+1)^\alpha} = \frac{\Gamma(m+1)}{\Gamma(m+1+\alpha)} k_m^* = k_m^\alpha \kappa_m^{**},$$

since the multipliers

$$k_m^* = \frac{\Gamma(m+1+\alpha)}{\Gamma(m+1)(m+1)^\alpha}, \quad k_m^{**} \frac{1}{(m+1)^\alpha k_m^\alpha}, \quad \frac{1}{k_m^*}, \quad \frac{1}{k_m^{**}}$$

satisfy (4.4). The relations $I_2^\alpha(X) = \mathbb{C} + I_j^\alpha(\overset{\circ}{X})$, $j = 3,4$, may be proved similarly. ∎

Lemma 4.2 enables us to use the operators $I_j^\alpha (j = 0,1,2,3,4)$ for the description of the space $X^\alpha(\Sigma_n)$. We shall not use the integral $I_4^\alpha\varphi$ for this purpose in the sequel because it is quite cumbersome.

The following theorem contains a description of the space $X^\alpha(\Sigma_n)$ for $0 < \alpha \leq n$ in terms of operator T_ε^α of the form (1.8).

THEOREM 4.1: *Let* $0 < \alpha \le n$, $f \in L_1(\Sigma_n)$.

I. *If* $1 \le p \le \infty$ *the following assertions are equivalent:*

a) $f \in X_p^\alpha(\Sigma_n)$;

b) *the sequence* $T_\varepsilon^\alpha f$ *converges in* X_p*-norm as* $\varepsilon \to 0$.

II. *If* $1 < p \le \infty$, *then* $f \in L_p^\alpha(\Sigma_n)$ *iff*

$$\sup_{0<\varepsilon<2} \|T_\varepsilon^\alpha f\|_p < \infty. \tag{4.5}$$

III. *The following assertions are equivalent:*

a′) $f \in M^\alpha(\Sigma_n)$;

b′) *the sequence* $\int_{\Sigma_n}(T_\varepsilon^\alpha f)(x)\omega(x)dx$ *converges as* $\varepsilon \to 0$ *for any* $\omega \in C(\Sigma_n)$;

c′)

$$\sup_{0<\varepsilon<2} \|T_\varepsilon^\alpha f\|_1 < \infty. \tag{4.6}$$

Proof: Let $f \in X_p^\alpha(\Sigma_n)$. Then $f = I^\alpha\varphi$, $\varphi \in X_p(\Sigma_n)$ (in the case $\alpha = n$ we mean $I^n\varphi$ to be a potential $I_\gamma^n\varphi$ of the form (2.1)), and $T_\varepsilon^\alpha f$ converges in X_p-norm by virtue of theorems 1.2, 2.1. Let us show that b) implies a). We note

$$I^\alpha T_\varepsilon^\alpha f = T_\varepsilon^\alpha I^\alpha f \tag{4.7}$$

(this equality may be easily verified on spherical harmonics, and then may be extended to $f \in X_p(\Sigma_n)$ by virtue of a boundedness of the operators I^α and T_ε^α in $X_p(\Sigma_n)$). If $0 < \alpha < n$, then, assuming $\varphi = \lim_{\varepsilon\to 0}^{(X_p)} T_\varepsilon^\alpha f$, with regard to (4.7) and to Theorem 1.2 we have

$$I^\alpha\varphi = \lim_{\varepsilon\to 0}^{(X_p)} I^\alpha T_\varepsilon^\alpha f = \lim_{\varepsilon\to 0}^{(X_p)} T_\varepsilon^\alpha I^\alpha f = f,$$

i.e., $f \in X_p^\alpha(\Sigma_n)$. In the case $\alpha = n$ we assume

$$\varphi = \frac{1}{\sigma_n k_0^n}\mu(f) + \lim_{\varepsilon\to 0}^{(X_p)} T_\varepsilon^n f, \qquad \mu(f) = \int_{\Sigma_n} f(y)dy,$$

and by virtue of (2.3) we have

$$I^n\varphi = \frac{\mu(f)}{\sigma_n k_0^n} I^n[1] + \lim_{\varepsilon\to 0}^{(X_p)} I^n T_\varepsilon^n f = \frac{\mu(f)}{\sigma_n} + f - \frac{1}{\sigma_n k_0^n}\mu(I^n f) = f,$$

i.e., $f \in X_p^n(\Sigma_n)$. To prove II let $f \in L_p^\alpha(\Sigma_n)$, $1 < p \le \infty$, i.e., $f = I^\alpha\varphi$, $\varphi \in L_p(\Sigma_n)$. If $0 < \alpha < n$, the inequality (4.5) follows from (1.25). If $\alpha = n$, then (4.5) is a consequence of both (2.6) and (2.7), since the convolution (2.8) satisfies (1.25) and $|k_{2,\varepsilon}| \le c_1$, $|k_{3,\varepsilon}(\tau)| \le c_2 h(\tau)$ (see(2.9)), with the constants c_1, c_2 not depending on $\varepsilon \in (0,1)$. Vice versa, since the unit ball in a space dual to a Banach space is compact in a weak* topology then by virtue of (4.5) there exists a sequence $\varepsilon_k \to 0$ and a function $\varphi \in L_p(\Sigma_n)$ such that

$$\lim_{\varepsilon_k\to 0}(T^\alpha_{\varepsilon_k} f,\omega) = (\varphi,\omega) \quad \forall\omega \in L_{p'}(\Sigma_n), \qquad \frac{1}{p'}+\frac{1}{p} = 1.$$

Hence

$$\begin{aligned}(I^\alpha\varphi,\omega) = (\varphi, I^\alpha\omega) &= \lim_{\varepsilon_k\to 0}(T^\alpha_{\varepsilon_k} f, I^\alpha\omega) =\\ &= \lim_{\varepsilon_k\to 0}(f, T^\alpha_{\varepsilon_k} I^\alpha\omega) = (f,\omega) \quad \forall\omega\in L_{p'}(\Sigma_n),\end{aligned}$$

i.e., $f = I^\alpha\varphi \in L_p^\alpha(\Sigma_n)$.

Let us prove III. If $f \in M^\alpha(\Sigma_n)$, then by virtue of Lemma 4.2 $f = I^\alpha\nu$, $\nu \in M(\Sigma_n)$, and b') follows from theorems 1.2, 2.1. Conversly, since the space $M(\Sigma_n)$ is weakly* complete, then there exists a measure $\nu \in M(\Sigma_n)$ such that $\lim_{\varepsilon\to 0}(T_\varepsilon^\alpha f,\omega) = (\nu,\omega)$ $\forall\omega \in C(\Sigma_n)$. Hence

(4.8)
$$(I^\alpha\nu,\omega) = (\nu, I^\alpha\omega) = \lim_{\varepsilon\to 0}(T_\varepsilon^\alpha f, I^\alpha\omega) = \lim_{\varepsilon\to 0}(f, T_\varepsilon^\alpha I^\alpha\omega) = (f,\omega)\ \forall\omega\in C(\Sigma_n),$$

and therefore $f = I^\alpha\nu \in M^\alpha(\Sigma_n)$. The proof of the equivalence of a$'$) and c$'$) is similar to the proof of the assertion II with replacing ε by $\varepsilon_k \to 0$ in (4.8). ∎

Let us exhibit a number of another description of spaces $L_p^\alpha(\Sigma_n)$, $C^\alpha(\Sigma_n)$, $M^\alpha(\Sigma_n)$ for all $\alpha > 0$ in terms of hypersingular constructions containing a Poisson integral. Given $\varepsilon \in (0,1)$, $\ell(\in \mathbb{N}) > \alpha$, we denote

$$(\mathcal{T}^\alpha_{0,\varepsilon} f)(x) = (\mathcal{T}^\alpha_\varepsilon f)(x)$$

(see (3.5)),

$$(4.9)\quad (\mathcal{T}^\alpha_{1,\varepsilon} f)(x) = \frac{1}{\kappa_\ell(\alpha)}\int_\varepsilon^\infty \eta^{-\alpha-1}\left[\sum_{j=0}^{\ell}\binom{\ell}{j}(-1)^j(1-j\eta)_+^\alpha f(x,1-j\eta)\right]d\eta,$$

$$(4.10)\quad (\mathcal{T}^\alpha_{2,\varepsilon} f(x) = \frac{1}{\kappa_\ell(\alpha)}\int_0^{1-\varepsilon}\left(\log\frac{1}{\rho}\right)^{-\alpha-1}\left[\sum_{j=0}^{\ell}\binom{\ell}{j}(-1)^j\rho^j f(x,\rho^j)\right]\frac{d\rho}{\rho},$$

$$(4.11)\qquad (\mathcal{T}^{\alpha}_{3,\varepsilon}f)(x)=\frac{1}{\kappa_\ell(\alpha)}\int_0^{1-\varepsilon}\left(\log\frac{1}{\rho}\right)^{-\alpha-1}\left[\sum_{j=0}^{\ell}\binom{\ell}{j}(-1)^j f(x,\rho^j)\right]\frac{d\rho}{\rho}.$$

The truncated hypersingular integrals (4.9)-(4.11) arise as $\mathcal{T}^{\alpha}_{\varepsilon}f$ when inverting the corresponding fractional integrals (8)-(10) in a formal way. Really, the Poisson integrals $(I^{\alpha}_j\varphi)(x,r)\quad (j=1,2,3)$ and $\varphi(x,r)$ are tied by means of the following fractional integrals:

$$(4.12)\qquad (I^{\alpha}_1\varphi)(x,r)=\frac{r^{-\alpha}}{\Gamma(\alpha)}\int_0^r (r-\rho)^{\alpha-1}\varphi(x,\rho)d\rho,$$

$$(4.13)\qquad (I^{\alpha}_2\varphi)(x,r)=\frac{r^{-1}}{\Gamma(\alpha)}\int_0^r\left(\log\frac{r}{\rho}\right)^{\alpha-1}\varphi(x,\rho)d\rho,$$

$$(4.14)\qquad (I^{\alpha}_3\varphi)(x,r)=\frac{1}{\Gamma(\alpha)}\int_0^r\left(\log\frac{r}{\rho}\right)^{\alpha-1}\varphi(x,\rho)\frac{d\rho}{\rho}.$$

The inversion of these integrals according to A. Marchaud's scheme leads to (4.9)-(4.11).

THEOREM 4.2: *Let $\alpha>0$, $f\in L_1(\Sigma_n)$; $j=0,1,2,3$.*

I. *If $1\le p\le\infty$ the following statements are equivalent:*

a) *$f\in X^{\alpha}_p(\Sigma_n)$;*

b) *the sequence $\mathcal{T}^{\alpha}_{j,\varepsilon}f$ converges as $\varepsilon\to 0$ in X_p-norm.*

II. *If $1<p\le\infty$, then $f\in L^{\alpha}_p(\Sigma_n)$ iff*

$$(4.15)\qquad \sup_{0<\varepsilon<1}\|\mathcal{T}^{\alpha}_{j,\varepsilon}f\|_p<\infty.$$

III. *The following statements are equivalent:*

a′) *$f\in M^{\alpha}(\Sigma_n)$;*

b′) *the sequence $\int_{\Sigma_n}(\mathcal{T}^{\alpha}_{j,\varepsilon}f)(x)\omega(x)dx$ converges as $\varepsilon\to 0$ for any $\omega\in C(\Sigma_n)$;*

c′)

$$(4.16)\qquad \sup_{0<\varepsilon<1}\|\mathcal{T}^{\alpha}_{j,\varepsilon}f\|_1<\infty.$$

Proof:

I. Let $f \in X_p^\alpha(\Sigma_n)$. Then $f = I_j^\alpha \varphi_j$, $\varphi_j \in X_p(\Sigma_n)$ $\forall j = 0,1,2$ and $f = I_3^\alpha \varphi_3 + c_0$, where $\varphi_3 \in \overset{\circ}{X}_p(\Sigma_n)$, $c_0 \in \mathbb{C}$. If $j = 0$, the sequence $\mathcal{T}_{0,\varepsilon}^\alpha f$ converges as $\varepsilon \to 0$ in X_p-norm due to Theorem 3.1. If $j = 1,2$, then using the argument as in the proof of Theorem 3.1 we obtain the representations

$$(4.17) \qquad (\mathcal{T}_{1,\varepsilon}^\alpha f)(x) = \int_0^\infty \lambda_{\ell,\alpha}(\eta)\varphi_1(x, 1-\varepsilon\eta)d\eta = (\Lambda_\varepsilon^{(1)}\varphi_1)(x),$$

$$(4.18) \qquad (\mathcal{T}_{2,\varepsilon}^\alpha f)(x) = \int_0^\infty \lambda_{\ell,\alpha}(\eta)(1-\varepsilon)^\eta \varphi_2(x,(1-\varepsilon)^\eta)d\eta = (\Lambda_\varepsilon^{(2)}\varepsilon_2)(x),$$

If $j = 3$, then $\mathcal{T}_{3,\varepsilon}^\alpha c_0 = 0$ and we have

$$(4.19) \qquad (\mathcal{T}_{3,\varepsilon}^\alpha f)(x) = \int_0^\infty \lambda_{\ell,\alpha(\eta)}\varphi_3(x,(1-\varepsilon)^\eta)d\eta = (\Lambda_\varepsilon^{(3)}\varphi_3)(x).$$

It follows from (4.17)-(4.19) that $\lim_{\varepsilon\to 0}^{(X_p)} \mathcal{T}_{j,\varepsilon}^\alpha f = \varphi_j$, $j = 1,2,3$. Conversly, let b) hold and $\varphi_j = \lim_{\varepsilon\to 0}^{(X_p)} \mathcal{T}_{j,\varepsilon}^\alpha f$. Then for $j = 1,2$ and in the case $j = 0$, $\frac{\alpha-n}{2} \notin \mathbb{Z}_+$ we obtain

$$(I_j^\alpha \varphi_j, \omega) = (\varphi_j, I_j^\alpha \omega) = \lim_{\varepsilon\to 0}(\mathcal{T}_{j,\varepsilon}^\alpha f, I_j^\alpha \omega) = \lim_{\varepsilon\to 0}(f, \mathcal{T}_{j,\varepsilon}^\alpha I_j^\alpha \omega) = (f,\omega)$$

for all $\omega \in S(\Sigma_n)$.

Hence $f = I_j^\alpha \varphi_j$ and therefore $f \in X_p^\alpha(\Sigma_n)$. Let $j = 0$, $\frac{\alpha-n}{2} = s \in \mathbb{Z}_+$. Then, using the notation and results from a previous section, for any $\omega \in S(\Sigma_n)$ we have

$$\begin{aligned}
(I^\alpha A_s\varphi_0, \omega) &= (I^\alpha A_s\varphi_0, A_s\omega) = (A_s\varphi_0, I^\alpha A_s\omega) \\
&= (\varphi_0, I^\alpha A_s\omega) = \lim_{\varepsilon\to 0}(\mathcal{T}_{0,\varepsilon}^\alpha f, I^\alpha A_s\omega) \\
&= \lim_{\varepsilon\to 0}(f, \mathcal{T}_{0,\varepsilon}^\alpha I^\alpha A_s\omega) = (f, A_s\omega) = (A_s f, \omega).
\end{aligned}$$

Hence

$$f = I^\alpha A_s\varphi + \sum_{m=0}^{s}\sum_{\mu} f_{m,\mu}Y_{m,\mu} \in X_p^\alpha(\Sigma_n).$$

If $j = 3$, then

$$f_\sigma = \frac{1}{\sigma_n}\int_{\Sigma_n} f(x)dx, \qquad f^0 = f - f_\sigma.$$

We note $(\mathcal{T}^{\alpha}_{3,\varepsilon}f)_\sigma = 0$, and therefore $(\varphi_3)_\sigma = 0$. But then $(I^\alpha_3\varphi_3)_\sigma = 0$, and for any $\omega \in S(\Sigma_n)$ we have

$$\begin{aligned}(I^\alpha_3\varphi_3, \omega) &= (I^\alpha_3\varphi_3, \omega^0) + (I^\alpha_3\varphi_3, \omega_\sigma)\\ &= (\varphi, I^\alpha_3\omega^0) = \lim_{\varepsilon\to 0}(\mathcal{T}^{\alpha}_{3,\varepsilon}f, I^\alpha_3\omega^0)\\ &= \lim_{\varepsilon\to 0}(f, \mathcal{T}^{\alpha}_{3,\varepsilon}I^\alpha_3\omega^0) = (f, \omega^0) = (f - f_\sigma, \omega),\end{aligned}$$

that gives $f = f_\sigma + I^\alpha_3\varphi_3$. Acoording to Lemma 4.2 this equality means that $f \in X^\alpha_p(\Sigma_n)$.

II. Let $f \in L^\alpha_p(\Sigma_n)$, $1 < p \leq \infty$. Then the estimate (4.15) for $j = 1,2,3$ follows from (4.17)-(4.19) due to properties of a Poisson integral with regard to (1.10). If $j = 0$, then according to (3.18), (3.17), (3.8) we obtain

$$(\mathcal{T}^{\alpha}_{0,\varepsilon}f)(x) = (\Lambda_\varepsilon A_s\varphi)(x) + \sum_{m=0}^{s}\sum_{\mu} f_{m,\mu}a^{\ell,\alpha}_m(\varepsilon)Y_{m,\mu}(x),$$

and the required estimate may be easily seen from the inequlity $\|(A_s\varphi)(\cdot,\rho)\|_p \leq \rho^{s+1}\|\varphi\|_p$. If $1 \leq p < \infty$ this inequality is a consequence of (3.12). In the case $p = \infty$ it follows from the estimate

$$\begin{aligned}&\left|\int_{\Sigma_n}\psi(x)(A_s\varphi)(x,\rho)dx\right| = \left|\int_{\Sigma_n}(A_s\psi)(x,\rho)\varphi(x)dx\right|\\ &\leq \|\varphi\|_\infty\|(A_s\psi)(\cdot,\rho)\|_1 \leq \rho^{s+1}\|\varphi\|_\infty\|\psi\|_1 \qquad \forall\psi \in L_1(\Sigma_n).\end{aligned}$$

The inverse assertion may be proved like the assertion "b)⇒a)" from the item I with replacing ε by ε_k (see the similar argument in the proof of the item II of Theorem 4.1).

III. Let $f \in M^\alpha(\Sigma_n)$. Then for $j = 0$ the assertion $b')$ follows from Lemma 4.2 and from Theorem 3.1. For $j = 1,2,3$ we replace the functions φ_j in (4.17)-(4.19) by measures $\nu_j \in M_j(\Sigma_n)$. Thus, $(\mathcal{T}^{\alpha}_{j,\varepsilon}f, \omega) = (\Lambda^{(j)}_\varepsilon\nu_j, \omega) = (\nu_j, \Lambda^{(j)}_\varepsilon\omega) \to (\nu_j, \omega)$ as $\varepsilon \to 0$ for any $\omega \in C(\Sigma_n)$. One can prove the assertion "b′) ⇒ a′)" by the same way as the asseriton "b) ⇒a)". The functions φ_j should be replaced by measures ν_j which are the weak* limits of sequences $\mathcal{T}^{\alpha}_{j,\varepsilon}f$. The equivalence of 1′) and 3′) may be proved similarly to the asserion II. ∎

Remark 4.1: While proving Theorem 4.2 we had obtained the inversion formulas for integrals I_j^α, $j = 1, 2, 3$. Namely, if $f = I_j^\alpha \varphi$, $\alpha > 0$, $1 \le p \le \infty$, $\varphi \in X_p(\Sigma_n)$ (if $j = 3$ we assume $\varphi \in \overset{\circ}{X}_p(\Sigma_n)$), then $\varphi = \mathcal{T}_j^\alpha f = \lim_{\varepsilon \to 0}^{(X_p)} \mathcal{T}_{j,\varepsilon}^\alpha f$. It is not hard to prove the a.e. convergence of $\mathcal{T}_{j,\varepsilon}^\alpha f$.

5. Integral equation with a power-logarithmic kernel

According to (3.4) there is an infinite number of ways to define a Riesz potential $I^\alpha \varphi$ for $\alpha = n+2k$, $k \in \mathbb{Z}_+$. Presently we restrict ourselves by the case when the Riesz potential of the order $\alpha = n + 2k$ is represented by a spherical convolution of the form

$$(5.1) \qquad (I_\gamma^{n+2k}\varphi)(x) = \gamma_{k,n} \int_{\Sigma_n} \varphi(y)|x-y|^{2k} \log \frac{\gamma}{|x-y|} dy,$$

where

$$\gamma_{k,n} = \frac{(-1)^k 2^{1-n-2k}}{\pi^{n/2} k! \Gamma(k+n/2)}.$$

The operator (5.1) is a generalization of the potential (2.1). It is easy to prove that

$$(5.2) \qquad (I_\gamma^{n+2k})(x) = \lim_{\alpha \to n+2k} \left[(I^\alpha \varphi(x) - c_{n,\alpha} \gamma^{\alpha-n-2k} \int_{\Sigma_n} \varphi(y)|x-y|^{2k} dy \right]$$

and

$$(5.3) \qquad (I_\gamma^{n+2k})(x) \sim \sum_{m,\mu} \mathcal{K}_{\gamma,k}(m) \varphi_{m,\mu} Y_{m,\mu}(x),$$

where

(5.4)

$$\mathcal{K}_{\gamma,\kappa}(m) = \begin{cases} \Gamma(m-k)/\Gamma(m+n+k) & \text{if } m > k, \\ \dfrac{(-1)^{k-m}}{(k-m)!(m+n+k-1)!}[\psi(m+n+k) - \psi(k+\frac{n}{2}) + & \\ +\psi(k-m+1) - \psi(k+1) + 2\log\frac{\gamma}{2}] & \text{if } m \le k. \end{cases}$$

As we saw in section 3, the invertibility of I_γ^{n+2k} depends on the equality $\mathcal{K}_{\gamma,k}(m) = 0$ for $m \le k$, i.e., it depends on γ.

LEMMA 5.1: *For any $\gamma > 0$ the equation $\mathcal{K}_{\gamma,k}(m) = 0$ has not more than one solution belonging to the set $\{0, 1, \ldots, k\}$. For a fixed $m \in \{0, 1, \ldots, k\}$ there exists one and only one $\gamma > 0$ such that $\mathcal{K}_{\gamma,k}(m) = 0$.*

Proof: Denote $u(z) = \psi(z+n+k) = \psi(k-z+1)$. According to formula 8.361(7) from [5] we have

$$u(z) = \int_0^1 \frac{2 - t^{z+n+k-1} - t^{k-z}}{1-t} dt - 2C,$$

C being an Euler constant. Since

$$\frac{du(z)}{dz} = \int_0^1 \frac{t^{k-z}(t^{2z+n-1} - 1)}{1-t} \log(1/t) dt < 0$$

for $0 \le z \le k$ then $u(z)$ is a strictly decreasing function, and therefore the equality $u(z) = \psi(k + \frac{n}{2}) + \psi(k+1) = 2\log\frac{2}{\gamma}$ with fixed $k \in \mathbb{Z}_+$ and $\gamma > 0$ is possible not more than for one $z \in [0, k]$. This gives the first assertion. The second one is obvious. ∎

Our results will be more attractive if we go over from (5.1) to the similar operator on a sphere $\Sigma_n(a) = \{x \in \mathbb{R}^{n+1} : |x| = a\}$. Let

$$(M_{a,k}\varphi)(x) = \gamma_{k,n} \int_{\Sigma_n(a)} \varphi(y)|x-y|^{2k} \log\frac{1}{|x-y|} dy. \tag{5.5}$$

An operator (5.5) may be called a Riesz potential of the order $\alpha = n + 2k$ on a sphere $\Sigma_n(a)$. For a function $f(x)$ given on $\Sigma_n(a)$ we denote $f_a(\xi) = f(a\xi)$, $\xi \in \Sigma_n$. Then $(M_{a,k}\varphi)_a(\xi) = a^{2k+n}(I_{1/a}^{n+2k}\varphi_a)(\xi)$. As we see below, the solvabiblity of the equation $M_{a,k}\varphi = f$ depends on the radius a.

Definition 5.1: The radius a in (5.5) will be called regular if $\mathcal{K}_{1/a,k}(m) \neq 0$ for all $m \in \{0, 1, \ldots, k\}$. If $\mathcal{K}_{1/a,k}(m) = 0$ for some $m \in \{0, 1, \ldots, k\}$ (by virtue of Lemma 5.1 such m is unique), then the radius a will be called a singular one of the type m.

For the convenience of the reader we remind some facts from the theory of Noether operators (see, e.g., [12]). Let X, Y be Banach spaces. A linear bounded operator $A : X \to Y$ is called a Noether operator if its range $A(X)$ is closed in Y and the numbers

$$\alpha(A) = \dim\ker A = \dim\{\varphi \in X : A\varphi = 0\},$$
$$\beta(A) = \dim\operatorname{coker} A = \dim Y/A(X)$$

are finite. The ordered pair $(\alpha(A), \beta(A))$ is called the d-characteristic of A. An operator R_ℓ (R_r) is said to be a left (right) regularizer of A if $R_\ell A = I_X + K_X$ $(AR_r = I_Y + K_Y)$, where I_X (I_Y) is an identity operator in X (in Y) and K_x (K_y) is a compact operator in X (in Y). If $R_\ell = R_r = R$, then the operator R is called a two-sided regularizer. A linear bounded operator A is a Noether operator iff it posseses both a left and a right bounded regularizers.

Assume

$$X_p(\Sigma_n(a)) = \begin{cases} L_p(\Sigma_n(a)) & \text{if } 1 \le p < \infty, \\ C(\Sigma_n(a)) & \text{if } p = \infty. \end{cases}$$

$X_p^\alpha(\Sigma_n(a))$ denotes a space of functions $f(x)$, $x \in \Sigma_n(a)$, for which $f_a(\xi) \in X_p^\alpha(\Sigma_n)$;

$$\|f\|_{X_p^\alpha(\Sigma_n(a))} \stackrel{\text{def}}{=} \|f_a\|_{X_p^\alpha(\Sigma_n)}.$$

THEOREM 5.1: *Let $1 \le p \le \infty$.*

I. *The operator $M_{a,k}$ acts as a bounded operator from $X_p(\Sigma_n(a))$ into*

$$X_p^{n+2k}(\Sigma_n(a)).$$

II. *If the radius a is regular, then the operator*

$$M_{a,k} : X_p(\Sigma_n(a)) \to X_p^{n+2k}(\Sigma_n(a))$$

is invertable, and a solution of the equation

$$M_{a,k}\varphi = f, \qquad f \in X_p^{n+2k}(\Sigma_n(a)) \tag{5.6}$$

has the following form

$$\varphi(x) = (T^{a,k} f)(x) + \sum_{j=0}^{k} \lambda_j \int_{\Sigma_n(a)} f(y) P_j^{(n/2-1, n/2-1)}\left(\frac{xy}{a^2}\right) dy, \tag{5.7}$$

where

$$(T^{a,k} f)(x) = \frac{1}{\kappa_\ell(n+2k)} \int_0^1 (a\eta)^{-n-2k} \left[\sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j (1-j\eta)_+^{n+k} f_a\left(\frac{x}{a}, 1-j\eta\right)\right] \frac{d\eta}{\eta}, \tag{5.8}$$

$$\lambda_j = \frac{a^{-2k-2n} j! d_n(j) \Gamma(n/2)}{\sigma_n \Gamma(j+n/2) \mathcal{K}_{1/a,k}(j)}.$$

III. *For every* $a > 0$ *the operator* $\mathcal{T}^{a,k}$ *annihilates on functions* $Y_{j,\mu}(x/a)$, $j \in \{0, 1, \ldots, k\}$, $\mu \in \{1, \ldots, d_n(j)\}$, *and acts as a bounded operator from* $X_p^{n+2k}(\Sigma_n(a))$ *into* $X_p(\Sigma_n(a))$.

IV. *If* a *is a singular radius of the type* m *(there exit exactly* $k+1$ *such radii!), then the operator:* $X_p(\Sigma_n(a)) \to X_p^{n+2k}(\Sigma_n(a))$ *is a Noether operator with the d-characteristic* $(d_n(m), d_n(m))$. *In this case the following statements hold:*

a) *The hypersingular operator* $\mathcal{T}^{a,\kappa}$ *(5.8) is a two-sided regularizer for* $M_{a,k}$.

b) *If the equation (5.6) is solvable, then its "general" solution has the form*

$$(5.9)\quad \varphi(x) = (\mathcal{T}^{a,k} f)(x) + \sum_{\substack{j=0 \\ (j \neq m)}}^{k} \lambda_j \int_{\Sigma_n(a)} f(y) P_j^{(n/2-1,\, n/2-1)}\Big(\frac{xy}{a^2}\Big) dy + \sum_{\mu=0}^{d_n(m)} c_\mu Y_{m,\mu}(x/a),$$

c_μ *being arbitrary constants.*

c) *The equation (5.6) is solvable in* $X_p(\Sigma_n(a))$ *iff*

$$(5.10)\qquad (f_a)_{m,\mu} = 0 \quad \forall \mu = 1, 2, \ldots, d_n(m).$$

Proof: The assertion I follows from Lemma 4.2. The assertion II follows from Lemma 4.2 and from Theorem 3.1. The formula (5.7) may be deduced from the addition theorem for spherical harmonics ([4]). The first assertion from III is obvious if we use the equality $(\mathcal{T}^{a,k} f)_a(\xi) = a^{-2k-n}(\mathcal{T}^{n+2k} f_a)(\xi)$ and Lemma 3.2. Let us prove that the operator $\mathcal{T}^{a,k}$ is bounded from $X_p^{n+2k}(\Sigma_n(a))$ into $X_p(\Sigma_n(a))$. Given $f \in X_p^{n+2k}(\Sigma_n(a))$ we have $\tilde f(\tilde x) = f_a(\tilde x/b) \in X_p^{n+2k}(\Sigma_n(b))$ for any $b > 0$. If we choose b regular, then according to II there is a function $\tilde\varphi(\tilde x) \in X_p(\Sigma_n(b))$ such that $\tilde f(\tilde x) = (M_{b,k}\tilde\varphi)(\tilde x)$. Assuming

$$\varphi(y) = (b/a)^{n+2k} \tilde\varphi(by/a) \in X_p(\Sigma_n(a)),$$

we obtain

$$f(x) = (M_{b,k}\tilde\varphi)\Big(\frac{b}{a}x\Big) = \gamma_{k,n} \int_{\Sigma_n(b)} |\frac{b}{a}x - \tilde y|^{2k} \log \frac{1}{|b/a - \tilde y|} \tilde\varphi(\tilde y) d\tilde y$$

$$= (M_{a,k}\varphi)(x) + \gamma_{k,n}(b/a)^{2k+n}\log(a/b)\int_{\Sigma_n(a)} |x-y|^{2k}\varphi(y)dy.$$

Hence

$$f(x) = (M_{a,k}\varphi)(x) + \sum_{j=0}^{k}\sum_{\mu=1}^{d_n(j)} c_j(\varphi_a)_{j,\mu}Y_{j,\mu}(x/a), \tag{5.11}$$

where c_j may be readily calculated by the Funk-Hecke theorem. We note that by virtue of (3.19)

$$(\mathcal{T}^{a,k}M_{a,k}\varphi)(x) = \varphi(x) - \sum_{j=0}^{k}\sum_{\mu=0}^{d_n(j)}(\varphi_a)_{j,\mu}Y_{j,\mu}(x/a) = \tag{5.12}$$

(5.13)
$$= \varphi(x) - \sum_{j=0}^{k}\alpha_j\int_{\Sigma_n(a)}\varphi(y)P_j^{(n/2-1,n/2-1)}\Big(\frac{xy}{a^2}\Big)dy, \quad \alpha_j = \frac{\Gamma(n/2)d_n(j)j!}{\sigma_n a^n\Gamma(j+n/2)}.$$

Let us apply the operator $\mathcal{T}^{a,k}$ to (5.11). Since $\mathcal{T}^{a,k}$ annihilates on function $Y_{j,\mu}(x/a)$, $j=0,1,\ldots,k$, by virtue of (5.13) we obtain

$$(\mathcal{T}^{a,k}f)(x) = \varphi(x) - \sum_{j=0}^{k}\alpha_j\int_{\Sigma_n(a)}\varphi(y)P_j^{(n/2-1,n/2-1)}\Big(\frac{xy}{a^2}\Big)dy.$$

Hence

$$\|\mathcal{T}^{a,k}f\|_{X_p(\Sigma_n(a))} \le c\|\varphi\|_{X_p(\Sigma_n(a))} \le c\|\tilde\varphi\|_{X_p(\Sigma_n(b))} \le c\|\tilde f\|_{X_p^{n+2k}(\Sigma_n(b))}$$
$$= c\|f\|_{X_p^{n+2k}(\Sigma_n(a))}$$

(c denotes different constants).

Let us prove IV. The statement a) follows from (5.12) since the finite-dimensional operator in the right-hand side is compact. Thus, $M_{a,k}$ is a Noether operator. Since $\mathcal{K}_{1/a,k}(m) = 0$ and $\mathcal{K}_{1/a,k}(m_1) \neq 0$ for any $m_1 \neq m$ then $\dim\ker M_{a,k} = d_n(m)$ and $\ker M_{a,k}$ consists of linear combinations of functions $Y_{m,\mu}(x/a)$, $\mu = 1,2,\ldots,d_n(m)$. With regard to (5.12) this gives b). The necessity of c) is obvious because $\mathcal{K}_{1/a,k}(m) = 0$. To prove the sufficiency we rewrite (5.11) in the form

$$f_a(\xi) = (M_{a,k}g)_a(\xi) + c_m\sum_{\mu=1}^{d_n(m)}(\varphi_a)_{m,\mu}Y_{m,\mu}(\xi), \tag{5.14}$$

where

$$(5.15)\qquad g_a(\xi) = \varphi_a(\xi) + \sum_{\substack{j=0 \\ (j \neq m)}}^{k} \sum_{\mu=1}^{d_n(j)} \frac{c_j}{\mathcal{K}_{1/a,k}(j)} (\varphi_a)_{j,\mu} Y_{j,\mu}(\xi) \in X_p(\Sigma_n).$$

Calculating the Fourier-Laplace coefficients of both sides of (5.14), by virtue of (5.10) we obtain $(\varphi_a)_{m,\mu} = 0$. Hence $f = M_{a,k}g$, i.e.the equation (5.6) is solvable in $X_p(\Sigma_n(a))$.

To end the proof we note that

$$\dim \operatorname{coker} M_{a,k} = \dim X_p^{n+2k}(\Sigma_n(a))/M_{a,k}(X_p(\Sigma_n(a))) = d_n(m).$$

This equality follows from (5.14). ∎

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